Dropping one candidate under Beatpath and Ranked Pairs

Charles T. Munger, Jr.*
(Dated: October 6, 2023)

We report for Beatpath and Ranked Pairs an exhaustive examination of how the winning candidate changes when one candidate is dropped, for initial numbers of candidates of 4 and 5, and report sampling results of what happens for initial numbers of candidates from 6 to 18. Consistent with all the searches is the observation that if under Beatpath there is a single rank order, and if when the winning candidate drops there is also a single rank order, then in the new rank order the candidate who formerly placed second must place above the candidate who formerly placed third; and therefore we add to the proof by M. Schulze that the candidate who placed second cannot become placed last, the observation that the candidate who placed third cannot become placed first. Other than those two excluded cases, for candidates numbering from 4 to 18 we find that a candidate who placed anywhere in the original rank order could be found to be placed anywhere in the new rank order; in particular the candidate who had placed last could come to be placed first, and the candidate who had placed second could come to be placed second-to-last. These results are compared to the known properties of Ranked Pairs, and their larger political significance discussed.

I. INTRODUCTION

The outcomes of the Beatpath [1] and the Ranked Pairs [2] election methods are examined, looking in particular at how a rank order can change when one candidate is dropped from a race. We will work with what we shall call the common case of a victory matrix: all the magnitudes of the elements above the diagonal are distinct; and none of them are zero. We recall the element in row j and column k of a victory matrix is the number of ballots on which candidate j is placed ahead of candidate k, minus the number on which k is placed ahead of j; and the diagonal elements are by convention set to 0.

Section II describes how we can assemble the rank order of a large election, under Ranked Pairs or under Beatpath, from the rank orders of component elections that each contain a cycle of all that component's candidates; and how for such a component we can sample all its possible elections, either exhaustively or by sampling. Section III describes the known properties of a rank order under Ranked Pairs, and the changes possible when either its winning or losing candidate is dropped. Section IV describes the known properties under Beatpath, and also the unproven properties that are consistent with the numerical explorations in this paper, and the significance of the properties if they could be proved; it serves as a summary of the computational results described in the following sections VI through XI. For completeness Section V describes what can happen with 3 candidates, when the Ranked Pairs and the Beatpath outcomes are identical. Sections VI and VII show the results of exhaustive searches of the results of all elections with respectively four and five candidates under Beatpath. Sections VIII, IX, and X show the results for six, seven, and eight candidates of a sampling of the elections possible under Beatpath. Section XI show the results from a cruder sampling of elections under Beatpath for a number of candidates between nine and eighteen. Section XII shows how to construct a common-case election for N candidates where the number of rank orders consistent with Beatpath equals $(N-2)!$. Section XIII discusses the larger political significance of our mathematical and computational results.

To complete the exposition, Appendix A 1 contains proofs of pertinent results concerning Ranked Pairs, and Appendix B offers an alternative proof of a special case of a theorem [3] of M. Schulze concerning Beatpath. Appendix C) contains tables of victory matrices that under Beatpath have interesting changes in the rank order when the winning candidate drops out.

II. COMPONENTS AND SAMPLES

Every victory matrix V in the common case reduces to one where the magnitude of the elements above the diagonal are the positive integers 1 to $K = N(N-1)/2$, where N is the number of candidates; one merely replaces in V the off-diagonal element with the lowest magnitude with an element of magnitude 1, then the element with the next lowest with an element of magnitude 2, and so on, while preserving the signs and keeping the matrix antisymmetric. These changes preserve election outcomes under both Ranked Pairs and Beatpath, because those outcomes depend on the magnitudes of the elements only through their order from large to small, and not through their absolute size.

All possible common-case elections can therefore be represented by a permutation of the digits 1 to K put into the K above-diagonal elements of V, with an arbitrary sign given to each of those elements; for a net of $2^K K!$ elections. Only certain of these prove to be the fundamental building blocks for the rest.

Associate with each pattern of signs a directed graph, where each candidate is represented by a node, and for each pair of distinct nodes j and k an arc runs from j

* charlestmungerjr@gmail.com

to k if $V_{jk} > 0$ (and no arc runs from any node directly back to itself). In graph theory these graphs are called *tournament* graphs. A tournament graph is *reducible* [4] if its nodes can be partitioned into two nonempty sets B and C, such that an arc runs from each node in B to each node in C. Equivalently, if it is possible to label the nodes so that the adjacency matrix for the graph takes the form

$$\begin{pmatrix} B & 1 \\ 0 & C \end{pmatrix} ,$$

where the matrix B and the matrix C are square, and the 1 and the 0 rectangular blocks in the upper right and lower left corners contain elements that are respectively all 1 or all 0. A tournament graphs for which no such partition of nodes is possible is *irreducible*.

In such a graph, either or both of the set B and C, each considered as a representing graph that involves only its set of nodes, may prove reducible; and the sets into which either splits might in turn prove reducible, and so on. This sequence of splits must eventually halt; suppose in our example it does after the set C has split into the two irreducible sets D and E, when the adjacency matrix would take the form

$$\begin{pmatrix} B & 1 & 1 \\ 0 & D & 1 \\ 0 & 0 & E \end{pmatrix} ,$$

where the dimensions of the rectangular blocks of 1's and 0's vary so as to match the sizes of the square blocks B, D, and E.

The significance of this result for elections is as follows. The candidates corresponding to the nodes in the set B form the *Smith set* of the election. Under a method that is both *Smith* and *independent of Smith-dominated alternatives* (or ISDA), and that yields a single rank order for any election, the j the candidates in B will form the first j candidates in the whole election, and the order among those j will be identical to the order as if only the candidates in B had ever run. Then the k candidates in D will form the next k candidates in the rank order for the whole election, and the order among those k candidates will be identical to the order as if only the candidates in D had ever run. And so on down the diagonal, until the rank order for the whole election is known.

If we imagine some candidate in any of the sets, say set D, were dropped from the election, the only change in the whole rank order is would be the replacement of the rank order of the candidates in D by the rank order for the election in which the one candidate from D had been dropped.

Ranked Pairs has the properties required [5], so what rank orders might appear, and how a rank order can change if a candidate is dropped, reduce to the problems of what rank orders can appear in elections corresponding to irreducible graphs, and how such a rank order can change if a candidate is dropped.

Beatpath also has the properties required [6], and so much the same structure; except that Beatpath can form (in the common case of a victory matrix and when the number of candidates is 4 or more) an election that provides not a total order of the candidates but a partial order; or to describe the same phenomenon in other language, the outcomes of each of the separate elections B or C or D may be consistent with more than one rank order. This changes things only slightly; the list of all the rank orders consistent with the whole election are all those made by any choice of the rank orders consistent with B, followed by any choice of those consistent with C, followed by any choice of those consistent with D. Equivalently, if the outcome of a Beatpath election is a partial order that is represented by a (poset) matrix P, where $P_{jk} = 1$ if candidate j must precede candidate k in the partial order and is zero otherwise, then in our sample election the partial order for the whole election can be assembled from the partial orders from its separate component elections as

$$\begin{pmatrix} P(B) & 1 & 1 \\ 0 & P(D) & 1 \\ 0 & 0 & P(E) \end{pmatrix} .$$

If we learn that the candidate ranked last of five can come to be ranked first if the winner drops, we can for example construct an election where dropping the fourth candidate causes the candidate in eighth place to become ranked third, merely by padding the five-candidate race with three more candidates all of whom dominate the other five. We can also pad this election with any number of candidates all eight candidates dominate, so we will know we can construct such an election for any larger number of candidates as well. Therefore under either Ranked Pairs or Beatpath to study how a candidate in position k of a rank order can change position if a candidate in position $j \neq k$ were to drop, all we need study are the outcomes of elections whose patterns of signs in their victory matrices correspond to tournament graphs that are irreducible.

These graphs have alternate and equivalent descriptions: they are [7] the tournament graphs that are *strongly connected*; and also [8] the tournament graphs that contain a *Hamiltonian cycle*.

Actually we need study even less; the assignment of real candidates to particular rows (or equivalently columns) of V are irrelevant and may be freely permuted, since we intend either to run through, or to sample uniformly, all the permutations of the magnitudes of the elements of V. We are therefore free to position the required Hamiltonian cycle by forcing the sign of the elements on the superdiagonal of V, as well as the sign of the element V_{N1}, to be positive. This convention fixes N signs, so the number of patterns of signs to study is now 2^{K-N}, not 2^K.

More generally, the freedom to permute the rows and columns of V allows us to study only patterns of signs that correspond to the set of tournament graphs that are not merely irreducible but also topologically distinct, that is, identical up to a relabeling of the nodes.

The numbers # of irreducible tournament graphs for given N forms sequence A051337 in the Online Encyclopedia of Integer Sequences [9]. We have then the following table.

N	2^K	2^{K-N}	#
1	...	...	...
2	...	...	...
3	8	1	1
4	64	4	1
5	1024	32	6
6	32768	512	35
7	2097152	16384	353
8	268435456	1048576	6008

Aside from making an exhaustive search for $N = 5$ practical, there is another advantage to using only the tournament graphs. One question we seek to answer by a random search is, "If the candidate ranked first in an election were to drop out, can the candidate who had been ranked last become ranked first?" A problem with looking for this by random searches among the 2^K arbitrary patterns of signs is that it could only occur in an irreducible graph, and as N increases such graphs become a small fraction of the 2^K. For $N = 8$ this fraction [10] is 0.119, and by $N = 15$ it is 0.0018; one would be spending most of one's computer time analyzing elections that had zero chance of including what was being looked for.

On my simple laptop [11] an exhaustive search of the elections of all patterns of signs indexed by #, and all permutations of the magnitudes, can be run for $N = 5$ overnight. For larger N we obtain results by sampling some number of random permutations of magnitudes for each of the # graphs for that N. That number # increases sharply with N, but all the graphs can be found in a few hours for N up to 8 using the following simple tricks. I am sure there is a more efficient way, but this suffices.

On an N by N matrix our restriction to include only isomorphically distinct tournament graphs lets us set the sign of all the elements on the superdiagonal, and the sign of the element $(N, 1)$, to be $+1$; this ensures that the resulting graph must have at least one cycle. The N elements on the super-superdigaonal, together with the elements $(N-1, 1)$ and $(N, 2)$, need not be chosen independently from all 2^N patterns; we need accept only one out of each of the N patterns that result from a repeated cyclic relabeling of candidates $1 \to 2$ and $2 \to 3$ and $\ldots$, and $N-1 \to N$ and $N \to 1$; this trick reduces the number of patterns of signs we need consider by a factor of roughly N. For each such pattern, there remain above the diagonal $N(N-5)/2$ signs not yet determined. For these we construct a graph for each possible choice of these signs.

The trouble now is that many of these patterns of signs construct graphs that are mutually isomorphic.

We remove these isomorphic duplicates in a two-step process. If A is an adjacency matrix of a graph, then A^q has as its element A^q_{jk} the number of directed paths that connect node j to node k using with exactly $q-1$ intermediate nodes that are not in general distinct (so $j14213k$ is legitimate list of nodes that connect node j to node k; and so is $j123j$). The sorted list of row sums of A^q forms a vector such that any adjacency matrices yielding different vectors represent graphs that are are isomorphically distinct; however, matrices yielding the same vector may represent graphs that are isomorphically distinct or isomorphically identical.

We run through all the graphs and sort them into smaller groups by the components of this vector using $q = 4$. An advantage of this simple scheme [12] is that this vector allows us to place a graph within its proper group, and to create a new group, without having to do work that scales as the square of the number of graphs. Once we have the groups complete, we run through each separate group using a binary comparison (the implementation nauty in Maple™) to remove the duplicates within each group. For each group this takes a number of comparisons that is the order of the number of elements in the group, times the number of topologically distinct graphs to be found within the group; though the binary comparison tends to be slow, overall this scheme is feasible for N up to 8 even on my laptop. One check is that the number of distinct graphs found for each matches the number in sequence A051337.

Once we have the isomorphically distinct tournament graphs to give the admissible pattern of signs, we still have the problem of testing all permutations of the magnitudes 1 to $K = N(N-1)/2$ that may appear above the diagonal of the victory matrix, if we wish to do an exhaustive search of all the elections in the common case. Unfortunately $K!$ grows rapidly with N; the following table shows that with an exhaustive search for $N = 5$ proving to require a run overnight, achieving an exhaustive search even for $N = 6$ is unfeasible.

N	#	$K!$	$\# \cdot K!$
1	1	...	...
2	0	...	...
3	1	$3! = 6$	6
4	1	$6! = 720$	720
5	6	$10! = 3\,628\,800$	21 772 800
6	35	$15! = 1\,307\,674\,368\,000$	45 768 602 880 000

Accordingly we give the results of exhaustive searches only for $N = 4$ and 5; and for $N = 6$, 7, and 8 we merely sample for each of the # patterns of signs a certain number of random permutations of the magnitudes.

For $N \geq 9$, we abandon even the technique of constructing patterns of signs that are topologically distinct. Instead we simply set the superdiagonal elements of V, and the element $V_{N,1}$, to be positive, so to ensure the matrix contains an N-cycle; and construct each new matrix to be examined by choosing a random pattern for the remaining signs above the diagonal, and a random permutation of the integers 1 to K for the magnitude of the elements above the diagonal. This will give a different sampling distribution over the cyclic tournament graphs than we used for $N = 6$ to 8; but if our only desire is to see if we can stumble on an example of some phenomenon, that difference is irrelevant; we need only be lucky.

III. KNOWN PROPERTIES OF RANKED PAIRS

Under Ranked Pairs and in the common case, when the winning candidate drops the order of the remaining candidates is unchanged (see Appendix A 4), so the candidate who had placed second now places first, the candidate who had placed third now places second, and so on. Similarly if the losing candidate drops, the order of the remaining candidates is likewise unchanged (see Appendix A 3), so the candidate who had placed first remains placed first, the candidate who had placed second remains placed second, and so on. This combination of properties is called *local stability* [13].

As a consequence of local stability, for any candidates a and b where a immediately precedes b in the rank order we must have $V_{ab} > 0$. That is, in an actual election, there will always be more ballots on which a was placed above b, than b placed above a. Or if we define a *break* in a rank order as a pair of candidates a and b who are adjacent in that rank order and for which for which $V_{ab} < 0$, no rank order from Ranked Pairs has a break.

The Ranked Pairs rank order therefore represents a Hamiltonian path in V. The Ranked Pairs algorithm in the common case can them be reinterpreted as follows. Start with the finite set of all the Hamiltonian paths of a victory matrix. Alternate repeatedly searching the set of paths for the largest element V_{ab} for any adjacent nodes a and b in any path, and then deleting from the set all paths that do not include that element. When one path is left, it will be the rank order of Ranked Pairs.

IV. KNOWN AND CONJECTURED PROPERTIES OF BEATPATH

For $N > 3$ our numerical explorations of Beatpath, so far as they have gone for tournament graphs that contain Hamiltonian cycles, and whose victory matrices are also in the common case, are consistent with those graphs conforming to the following properties.

(1) Though the Beatpath rank order is not always unique, there is always a unique winner.

(2) When the Beatpath rank order and the rank order when the winning candidate is dropped are both unique, then

(a) it is impossible for the candidate who had been ranked second to become ranked last.

(b) it is impossible for the candidate who had been ranked third to become ranked first.

(c) it is impossible for the candidate who had been ranked second to place behind the candidate who had been ranked third.

(d) with the exceptions of the cases covered by (a) and (b), a not-winning candidate in any position in the original race can come to occupy any position in the race after the winning candidate is dropped.

(3) Among the elections for which the Beatpath rank order is unique can be found a rank order where a break occurs at any desired position.

Of these, the only property that has, to my knowledge, been proved true is (2a), as a consequence of the following more general result [3] proved by M. Schulze, which we state as follows:

> Suppose there is a unique winner in an N-candidate election. If that winning candidate drops, no candidate who lost only to that winning candidate can finish as the sole candidate in last place.

An alternative and derivative but shorter proof of the more limited proposition (2a) is given in Appendix B.

Note that (2c) is a more general property that would imply both (2a) and (2b).

Examples for every case of (2d) have been found by computer search to exist for all N from 3 to 18; it is therefore known to be true for $N \leq 18$.

Otherwise exhaustive examination of all possible common-case, tournament elections with cycles have found all these properties hold for $N = 4$ and $N = 5$.

Beatpath, like Ranked Pairs, has inversion symmetry [14]: if all the elements of any victory matrix change sign, the partial order inverts also, in the sense that if candidate j was required to precede candidate k in the original partial order, then candidate k is required to precede candidate j in the new one. Each of the properties (1) through (2d) above is therefore true if and only if the following corresponding property is also true:

(1') Though the Beatpath rank order is not always unique, there is always a unique loser.

(2') When the Beatpath rank order and the rank order when the losing candidate is dropped are both unique, then

(a') it is impossible for the candidate who had been ranked second to last to become ranked first.

(b') it is impossible for the candidate who had been ranked third-to-last to become ranked last.

(c') it is impossible for the candidate who had been ranked second-to-last to place ahead of the candidate who had been ranked third-to-last.

(d') with the exceptions of the cases covered by (a') and (b'), a not-losing candidate in any position in the original race can come to occupy any position in the race after the losing candidate is dropped.

What might be gained should these properties by proved true?

Property (1) would allow the same tiebreaking scheme used for Ranked Pairs (in case any of the elements of the victory matrix are tied, or are zero) also to give a unique winner under Beatpath, and so provide yet another way to resolve ties in Beatpath to give a unique winner for every possible election. Also a proof of property (1), together with the construction in Section XII, would answer the question of how many rank orders can be consistent with the partial order provided by Beatpath.

Property (2c) would establish a basic structural property of Beatpath: one could not by dropping the candidate placed first move the candidate placed second below the candidate placed third. That would include the already-established (2a), and also establish (2b). It would have a curious practical consequence: if under Beatpath the winning candidate is dropped, and for some reason the candidate who was placed second is not promoted to be the winner, the candidate who was placed third could never be promoted to be the winner—only candidates behind him in the rank order could be eligible.

Exhaustive search shows that all the properties hold for $N = 4$ and $N = 5$, and so at least describe Beatpath results in the general election in a "top-four" or "top-five" system of elections; which might already be adequate for actual practice. Even should one or more of these properties fail for some larger value of N, it would be useful to know what it was. As it is, the following numerical explorations show that any violation of the conjectures has to be rare: assuming an N-candidate election manifests a cycle involving all N candidates, the probability of a violation has to be less than about 1 in a million elections.

We now look at the results of our searches in detail in sections V through XI for successively larger numbers of candidates.

V. THREE CANDIDATES

Beatpath and Ranked Pairs give identical outcomes for $N \leq 3$.

There is only one cyclic tournament graph for $N = 3$, which is shown in Figure 1. Any given election with this graph will have only one of the three rank orders $[1, 2, 3]$, $[2, 3, 1]$, or $[3, 1, 2]$, but without more information than the signs of the elements of V we cannot say which. Suppose the rank order happens to be $[1, 2, 3]$. Then if 1 drops, the rank order will become as desired $[2, 3]$; and if 3 drops, the rank order will become as desired $[1, 2]$; but if 2 drops, the rank order never becomes $[1, 3]$ but reverses, becoming $[3, 1]$.

Any election method that produces a single rank order of the candidates must contain a pair of candidates such that candidate a is placed above b in that rank order, and yet here $V_{ab} < 0$; then if the third candidate c drops out of the race, the new rank order must place b above a. In the case of Beatpath and Ranked Pairs, this phenomenon takes a common form, which happens to be that if the candidate placed second drops, the candidate ranked first becomes ranked last, and the candidate ranked last becomes ranked first.

VI. FOUR CANDIDATES

Here too there is only one cyclic tournament graph, that of Figure 2. All four candidates are in a 4-cycle of preferences; the corresponding victory matrix has the pattern of signs

$$\begin{matrix} \cdots & + & + & - \\ - & \cdots & + & + \\ - & - & \cdots & + \\ + & - & - & \cdots \end{matrix}$$

For any N, Ranked Pairs always gives a total order of the candidates and so a unique rank order. Beginning with $N = 4$, Beatpath does not always give a total order.

Empirically, for $N = 4$ it is found that when the Beatpath rank order is not unique, there are two rank orders consistent with the partial order; the candidate in first place in each is the same; and also that the candidate found in last place in each is the same. Those facts are related because Beatpath has inversion symmetry, so when V changes sign the in the partial order reverses. So if for any N the candidates found in first place in all the multiple rank orders match, then the candidates found in last place in all the multiple rank orders must also match.

In 696 of the 720 elections, Beatpath yields a unique rank order, and for these we find

$$B'_{jk} = \begin{bmatrix} \cdots & 672 & 0 & 24 \\ 288 & \cdots & 384 & 24 \\ 480 & 0 & \cdots & 216 \\ 600 & 96 & 0 & \cdots \end{bmatrix},$$

where B'_{jk} is the number of elections in which if the candidate in position j of the 4-candidate rank order drops out, in the resulting 3-candidate rank order the candidate in position k of the 4-candidate rank order becomes the winner.

If a candidate drops out, he cannot place anywhere in the resulting smaller election, so while the diagonal elements of B' are all zero they are indicated by ellipses because they are meaningless.

For the other 24 races have a partial order consistent with two rank orders, $[1, 2, 3, 4]$ and $[1, 3, 2, 4]$, so there is no doubt that candidate 1 loses and candidate 4 is last; but there is no determining whether it is candidate 2 or candidate 3 who would be in second place. For these 24 the ordinary expectations are met:

If the candidate in first place drops, the new winner is always one of the candidates in original second or third place; the other is second, and the original loser is third.

If one of the tied candidates 2 or 3 drops, the new winner is always the original winner; the survivor of the tied candidates is second, and the original loser is third.

If the candidate in fourth place drops, the new winner is always the original winner, and the tied candidates follow as second and third.

Roughly speaking, these facts suggest we should say that for these 24 cases the matrix that corresponds to B'_{jk} may be taken to be

$$B''_{jk} = \begin{bmatrix} \cdots & 24 & 0 & 0 \\ 24 & \cdots & 0 & 0 \\ 24 & 0 & \cdots & 0 \\ 24 & 0 & 0 & \cdots \end{bmatrix} ;$$

that is, we either elect the original winning candidate, or if he is the one to drop, a candidate originally tied for second. If we do this, for the 720 elections in total we would have

$$B = B' + B'' = \begin{bmatrix} \cdots & 696 & 0 & 24 \\ 312 & \cdots & 384 & 24 \\ 504 & 0 & \cdots & 216 \\ 624 & 96 & 0 & \cdots \end{bmatrix} .$$

For Ranked Pairs the tabulation is without ambiguity, because it always yields a single rank order without ties; and the corresponding matrix is

$$R = \begin{bmatrix} \cdots & 720 & 0 & 0 \\ 336 & \cdots & 336 & 48 \\ 432 & 48 & \cdots & 240 \\ 720 & 0 & 0 & \cdots \end{bmatrix} .$$

Ranked Pairs is leader drop-steady: if one drops the winner, the remaining rank order doesn't change, so naturally since in row 1 we are dropping the winning candidate, the second candidate must win. Hence $R_{1,2} = 720$, and the other zeros in that row are explained. Ranked Pairs is also trailer drop-steady; if one drops the loser, the remaining rank order doesn't change, so naturally since in row 4 we are dropping the losing candidate, the original first candidate must win. Hence $R_{4,1} = 720$, and the other zeros in that row are explained. So no zeros have been found that are not tied to some deeper property of Ranked Pairs.

Overall, which pattern for 4 candidates, that of Beatpath or of Ranked Pairs, could be said to be better? Suppose for the sake of argument that it is equally likely that any one of the four candidates will drop. The natural candidate to be elected is then the one in first place, if he is not dropped; otherwise the candidate in second place. There are $4 \times 720 = 2880$ cases of dropping a candidate; Beatpath departs from electing the natural candidate in

$$24 + (384 + 24) + 216 + 96 = 744$$

cases. Ranked Pairs departs from electing from electing the natural candidate in

$$0 + (336 + 48) + (48 + 240) + 0 = 672$$

cases. Numerically that is not enough of a difference to care about, though it is marginally in favor of Ranked Pairs.

Now suppose we try a weighting; suppose is not merely the fact of a failure to elect the natural candidate, but we should weight each by the difference between what happened and electing the natural candidate; so electing a second-place candidate instead of a first costs 1, but electing a fourth-place candidate when electing a first is possible (that is, we didn't drop the first) will cost 3, and electing a fourth-place candidate when electing a second is possible (that is, we dropped the first) will cost 2. Then the weighted cost of Beatpath is

$$(24 \times 2) + (384 \times 2) + (24 \times 3) + (216 \times 3) + (96 \times 1) = 1632$$

while the weighted cost of Ranked Pairs is

$$(336 \times 2) + (48 \times 3) + (48 \times 1) + (240 \times 3) = 1584 ,$$

which numerically is also not enough of a difference to care about, though it too is marginally in favor of Ranked Pairs.

If what is shocking is the mere fact that the candidate ranked fourth could be elected at all, then Beatpath is better, since the number of cases where that occurs is

$$24 + 24 + 216 = 264 ,$$

while for Ranked Pairs it is

$$48 + 240 = 288 .$$

If we weight, under Beatpath the harm is

$$2 \times 24 + 3 \times 24 + 3 \times 216 = 768 ,$$

while for Ranked Pairs the harm is

$$2 \times 0 + 3 \times 48 + 3 \times 240 = 864 .$$

Both of these differences too are not enough to care about, though both are marginally in favor of Beatpath.

A. A deeper look at Beatpath for $N = 4$

For the 696 elections where for $N = 4$ Beatpath yields a single rank order, we can look at where each candidate places depending on which candidate is dropped. Let M_{nm} indicate the number of times the candidate in position m of the 4-candidate rank order winds up in a particular place in the three-candidate-rank order, when the candidate in position n of the 4-candidate rank order drops. The various matrices M are

$$\begin{array}{rrrr} \cdots & 672 & 0_a & 24 \\ 288 & \cdots & 384 & 24 \\ 480 & 0_b & \cdots & 216 \\ 600 & 96 & 0_c & \cdots \end{array} \qquad \text{to place \#1 of three}$$

$$\begin{array}{rrrr} \cdots & 24 & 600 & 72 \\ 192 & \cdots & 312 & 192 \\ 192 & 312 & \cdots & 192 \\ 72 & 600 & 24 & \cdots \end{array} \qquad \text{to place \#2 of three}$$

$$\begin{array}{cccc} \cdots & 0_c & 96 & 600 \\ 216 & \cdots & 0_b & 480 \\ 24 & 384 & \cdots & 288 \\ 24 & 0_a & 672 & \cdots \end{array} \quad \text{to place \#3 of three}$$

The first matrix is a repeat of B'.

These matrices have manifest symmetries when the position of the elements are inverted with respect to a point in the center of the matrix: the first becomes the third, the third becomes the first, and the second matrix returns to itself. These symmetries result from Beatpath, like Ranked Pairs, having inversion symmetry: if we change the sign of the victory matrix V, the rank order under Beatpath reverses.

For example, consider the 0_c in position $1,2$ of the third matrix, which says that there were zero examples when dropping the candidate in first place in a 4-candidate race make the candidate in second place appear last in the resulting 3-candidate race. Suppose there were an example, when for some V and for candidates A, B, C and D we had a change in rank order such as

$$[A,B,C,D] \to [C,D,B] \ .$$

Changing the sign of V reverses the rank order of both elections, and so for $-V$ we would have

$$[D,C,B,A] \to [B,D,C] \ .$$

But this is an election where dropping the candidate who placed last makes the candidate who placed third appear first in the resulting 3-candidate race. And therefore there would also have to be an example in position $4,3$ of the first matrix, as well. Thus the zeros marked 0_c in the first and third matrices are related. Similar arguments relate the pair of zeros marked 0_a, and the pair of zeros marked 0_b.

Separate testing all of the cases when the first candidate drops shows that in the resulting three-candidate election the candidate who had placed second has to continue to precede the candidate who had placed third.

From this more general property, the zero 0_a in position $(1,3)$ of the first matrix follows—of course, the candidate who ranked third could never be promoted to first, because he would then place ahead of all the remaining candidates, including the candidate who had placed second. It would also follow that if we dropped the first candidate, that the candidate who had placed second can never become ranked last, because he would have to continue to precede the candidate who had placed third, and so the 0_c in position $(1,2)$ of the third matrix would follow.

The zeros θ_a and θ_c have their analogs for elections with more candidates; but the zeros θ_b do not. That for $N=4$ it is true that if the candidate placed third drops, the candidate who had been placed second is never promoted to being ranked first, such a restriction does not appear for $N=5$.

The second matrix shows that any of the surviving candidates can come to occupy second place, whichever of the candidates in the 4-candidate race one drops.

M. Schulze has shown that property tied to the zero θ_c generalizes. The part of his result [3] relevant here is that for any N, an election with a single rank order cannot, if the candidate placed first drops, give a single rank order with the candidate who had placed second now in last place.

For $N=4$ a sample of matrices that when the winning candidate drops, put each of the remaining candidates into the new rank order in each of the places possible, appear in Appendix C.

VII. FIVE CANDIDATES

There are six distinct cyclic tournament graphs that contain a 5-cycle, which are shown in Figures 3 through 8. As it happens, if we run through all $10! = 3628800$ permutations of the 10 distinct magnitudes of the above-diagonal elements for each of these six, it proves that under Beatpath there is, as there is under Ranked Pairs, always a unique winner, though the partial order provided by Beatpath can be consistent with more than one rank order. Running Beatpath to determine the outcomes for all elections took $34,000\,\mathrm{s}$ on my laptop.

For none of the six graphs is having 7 or more rank orders possible; that 6 can be achieved is shown by the construction in Section XII. The complete breakdown of the number of rank orders that are consistent, for each of the graphs in Figures 3 through 8 in order by row, are

	1	2	3	4	5	6
Fig. 3	3274559	285120	0	64800	0	4320
Fig. 4	3496319	118080	0	14400	0	0
Fig. 5	3386879	213120	0	28800	0	0
Fig. 6	3481919	69120	0	73440	0	4320
Fig. 7	3496319	112320	0	20160	0	0
Fig. 8	3427199	187200	0	14400	0	0

Since for 4 and for 5 candidates there is always a unique winner (and by inversion symmetry, always a unique loser), we may examine all the elections to see in how many elections, if the candidate ranked first among the five drops, the candidate ranked last among the five is promoted to be first among the remaining four. The totals are

3	52592
4	31840
5	64116
6	62436
7	47308
8	74180

of the 3628800 elections per graph.

We can also pose the following question. For each of the six graphs, if the five-candidate election yields a single rank order, and if when the candidate who placed first drops, the four-candidate election that results also yields a single rank order, how many times does a candidate who placed first through fifth in the five-candidate race appears in each of the four positions first through fourth in the four-candidate race? Obviously the candidate dropped places in any position 0 times; we will indicate such trivial zeros by ellipses. We find

graph 3, sample size 3274559

$$\begin{bmatrix} \cdots & \cdots & \cdots & \cdots \\ 3160719 & 94400 & 19440 & 0 \\ 0 & 2941719 & 163160 & 169680 \\ 64848 & 132672 & 2739303 & 337736 \\ 48992 & 105768 & 352656 & 2767143 \end{bmatrix}$$

graph 4, sample size 3474719

$$\begin{bmatrix} \cdots & \cdots & \cdots & \cdots \\ 3389759 & 69440 & 15520 & 0 \\ 0 & 3213119 & 155600 & 106000 \\ 53120 & 123760 & 3097559 & 200280 \\ 31840 & 68400 & 206040 & 3168439 \end{bmatrix}$$

graph 5, sample size 3354031

$$\begin{bmatrix} \cdots & \cdots & \cdots & \cdots \\ 3215219 & 131308 & 7504 & 0 \\ 0 & 2976175 & 283968 & 93888 \\ 76400 & 176144 & 2875395 & 226902 \\ 62412 & 70404 & 187164 & 3034051 \end{bmatrix}$$

graph 6, sample size 3481919

$$\begin{bmatrix} \cdots & \cdots & \cdots & \cdots \\ 3364019 & 89144 & 28656 & 0 \\ 0 & 3199559 & 107784 & 174576 \\ 59136 & 135336 & 3048323 & 239124 \\ 58764 & 57780 & 297156 & 3068219 \end{bmatrix}$$

graph 7, sample size 3463223

$$\begin{bmatrix} \cdots & \cdots & \cdots & \cdots \\ 3343163 & 103164 & 16896 & 0 \\ 0 & 3148223 & 198256 & 116744 \\ 73376 & 158200 & 3056451 & 175196 \\ 46684 & 53636 & 191620 & 3171283 \end{bmatrix}$$

graph 8, sample size 3369719

$$\begin{bmatrix} \cdots & \cdots & \cdots & \cdots \\ 3210099 & 151300 & 8320 & 0 \\ 0 & 2994879 & 266640 & 108200 \\ 854440 & 171360 & 2905739 & 207180 \\ 74180 & 52180 & 189020 & 3054339 \end{bmatrix}$$

The number of elections sampled for each of the six graphs in such a sample varies, because for each pattern of signs the number of elections that give a unique rank order for the five candidates, and a unique rank order when the winning candidate is dropped, happens to vary.

The tableaux for the different graphs for $N = 5$ manifest no individual structures, so we sum these cases into a single array of all 20 418 176 elections where a single rank order results both in the original election and the election when the winning candidate is dropped:

all graphs 3 through 8: $B^{(5)} =$

$$\begin{bmatrix} \cdots & \cdots & \cdots & \cdots \\ 19682984 & 638856 & 96336 & 0 \\ 0 & 18473680 & 1175408 & 769088 \\ 412320 & 897472 & 17722776 & 1385608 \\ 322872 & 408168 & 1423656 & 18263480 \end{bmatrix}$$

That in these samples it is impossible to have the second of the five candidate emerge as the fourth of the four is expected from Schulze's theorem. That in these samples it is impossible for the third of the five candidates to emerge as the first of the four may be new.

Both follow from the following more general property, which exhaustive search showed is true in every common-case election for $n = 5$: when the rank order for the five candidates is unique, and the rank order when the winning candidate is dropped is unique, the candidate who had placed second always precedes in the new rank order the candidate who had placed third.

We note that aside from that constraint, when the candidate who placed first is dropped, any of the candidates not dropped appear anywhere in the new rank order; in particular the candidate originally ranked fifth and last can be promoted to be ranked first.

That being so, the most extreme change would be to have the candidate who was ranked last promoted to being ranked first, while simultaneously having the candidate who was ranked second demoted to being ranked third, and the candidate who was ranked third demoted to being ranked fourth and last; that is, to have a rank order 1 2 3 4 5 change to 5 4 2 3. However exhaustive search proves this to be impossible in the common case for $N = 5$.

For breaks in the Beatpath rank order, the smallest possible number of breaks is 0, and the largest possible (whether or not it can be achieved) is $N - 1$. The distribution of the number of breaks from 0 to 4 is

$$[12\,929\,760,\ 6\,394\,560,\ 1\,238\,880,\ 0,\ 0]\ ,$$

and the number of times a break in each position from $j = 1$ to 4 occurred has the distribution

$$[3\,289\,920,\ 1\,146\,240,\ 1\,146\,240,\ 3\,289\,920]\ .$$

Breaks may therefore occur anywhere in a rank order; and we note a substantial fraction

$$3\,289\,920/20\,418\,176 = 16.1\%$$

occur annoyingly between the candidates placed first and second, where they are sure to occasion controversy, as the supporters of the candidate placed second point out that a majority of the voters prefer him to the candidate who placed first.

For $N = 5$ a sample of matrices that when the winning candidate drops, put each of the remainging candidates into the new rank order in each of the places possible, appear in Appendix C.

VIII. SIX CANDIDATES

We run 200 000 permutations of the magnitudes of the elements of V for each of the 35 patterns of signs, and so 7 500 000 elections under Beatpath; this many elections took 16,000 s to analyse. All the elections that had multiple rank orders yielded both a single winning and a single losing candidate. This being so, the largest number of rank orders that could be found, according to the arguments in Section XII, is $4! = 24$; elections were found with that number, but none with more.

Of the 7 500 000 elections, 6 413 133 yielded a single election rank order. Define a break as in Section VII; the distribution of these elections over the number of breaks, from 0 to 5, was

$$[3\,231\,183,\ 2\,382\,843,\ 715\,641,\ 83\,466,\ 0,\ 0]\ ,$$

so that the largest number of breaks found was 3. Also of interest is the distribution of the locations of the breaks; evidently we can have multiple breaks in any one rank order. Let us count the number of times a break occurs between the candidates who placed as j and as $j + 1$ in the rank order, for j ranging from 1 to $N - 1$. The number of occurrences of each kind of break was

$$[1\,190\,162,\ 610\,871,\ 463\,028,\ 611\,621,\ 1\,188\,841]\ .$$

So breaks may occur anywhere; and a substantial fraction

$$1\,190\,162/6\,413\,133 = 18.8\%$$

occur annoyingly between the candidates placed first and second, where they are sure to occasion controversy.

Of the elections studied 6 342 540 both yield a single rank order and, when the winning candidate is dropped, have the resulting election also yield a single rank order. In none of these did the candidate who originally placed third precede the candidate who originally placed second. From this observation we can conclude that in our sample the candidate who originally placed third cannot become placed first, and the candidate who originally placed second cannot become placed last. The new position of the candidate who had placed third, minus the new position of the candidate who had placed second, runs from 1 to a maximum of 4; and the distribution of elections over those values runs

$$[6\,034\,495,\ 164\,857,\ 81\,141,\ 62\,047]\ .$$

So while the candidate who had placed third must continue to be placed behind the candidate who had placed second, he may appear anywhere behind.

In our sample of elections, 6 342 540 had both a unique rank order and, when the winning candidate was dropped, that election too had a unique rank order. The following table $B^{(6)}_{jk}$ indicates, for the candidate who placed in position j in the original election, the position k in the second election where that same candidate placed; j runs 1 to 6, and k from 1 to 5.

$$B^{(6)} = \begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ 6135468 & 170359 & 28798 & 7915 & 0 \\ 0 & 5859509 & 289994 & 104125 & 88912 \\ 84132 & 171358 & 5715280 & 254149 & 117621 \\ 66598 & 88029 & 232177 & 5652882 & 302854 \\ 56342 & 53285 & 76291 & 323469 & 5833153 \end{bmatrix}$$

For $N = 6$ a sample of matrices that when the winning candidate drops, put each of the remaining candidates into the new rank order in each of the places possible, appear in Appendix C.

IX. SEVEN CANDIDATES

We run 20 000 permutations of the magnitudes of the elements of V for each of the 353 patterns of signs, and so 7 060 000 elections under Beatpath; this many elections took 22,000 s to analyze. All the elections that yield multiple rank orders yielded both a single winning candidate and a single losing candidate. That being so, the largest number of rank orders that could be found, according to the arguments in Section XII, is $5! = 120$; elections were found with a number of rank orders equal to 60, but none with more; a cautionary result, showing that because a random search fails to find an example of case, that does not mean it is not there to be found.

Of the 7 530 000 elections, 6 267 735 yielded a single rank order. Define a break as in Section VII; the distribution of these elections over the number of breaks, from 0 to 6, was

$$[2\,443\,302,\ 2\,508\,429,\ 1\,076\,332,\ 220\,876\ ,18\,796,\ 0,\ 0]\ ,$$

so that the largest number of breaks found was 4. Also as in Section VII of interest is the distribution of the locations of the breaks; the number of elections that had a break in any of the six possible places from 1 to 6 was

$$[1\,294\,551,\ 816\,145,\ 588\,857,\ 588\,137,\ 816\,302,\ 1\,294\,913]\ .$$

So breaks may occur anywhere; and a substantial fraction

$$1\,294\,551/6\,267\,735 = 20.7\%$$

occur annoyingly between the candidates placed first and second, where they are sure to occasion controversy.

Of the elections studied 6 180 575 both yield a single rank order and, when the winning candidate is dropped, have the resulting election also yield a single rank order. In none of these did the candidate who originally placed third precede the candidate who originally placed second.

It follows from this observation that in our sample the candidate who originally placed third cannot become placed first, and that the candidate originally placed second cannot become placed last. The new position of the candidate who had placed third, minus the new position of the candidate who had placed second, runs from 1 to a maximum of 5; and the distribution of elections over those values runs

$$[5\,948\,981,\ 119\,441,\ 51\,222,\ 32\,829,\ 28\,102]\ .$$

So while the candidate who had placed third must continue to be placed behind the candidate who had placed second, he may appear anywhere behind.

In our sample of elections, 6 180 575 had both a unique rank order and, when the winning candidate was dropped, that election too had a unique rank order. The following table $B^{(6)}_{jk}$ indicates, for the candidate who placed in position j in the original election, the position k in the second election where that same candidate placed; j runs 1 to 7, and k from 1 to 6.

$$B^{(7)} =$$
$$\begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 5997358 & 147452 & 25020 & 7882 & 2863 & 0 \\ 0 & 5791650 & 234734 & 72011 & 42106 & 40074 \\ 57574 & 113714 & 5693245 & 209846 & 63219 & 42977 \\ 44324 & 59278 & 147658 & 5655205 & 197996 & 76114 \\ 44133 & 41673 & 54900 & 186674 & 5634775 & 218420 \\ 37186 & 26808 & 25018 & 48957 & 239616 & 5802990 \end{bmatrix}$$

For $N = 7$ a sample of matrices that when the winning candidate drops, put each of the remaingfng candidates into the new rank order in each of the places possible, appear in Appendix C.

X. EIGHT CANDIDATES

We run 1000 permutations of the magnitudes of the elements of V for each of the 6008 patterns of signs, and so 6 008 000 elections under Beatpath; this many elections took $26,000$ s to analyze. All the elections that yield multiple rank orders yielded both a single winning candidate and a single losing candidate. That being so, the largest number of rank orders that could be found, according to the arguments in Section XII, is $6! = 720$; but none such were found in the sample

Of the 6 008 000 elections, 5 154 026 yielded a single rank order. Define a break as in Section VII; the distribution of these elections over the number of breaks, from 0 to 7, was

$$[1\,507\,417,\ 2\,050\,196,\ 1\,181\,668,\ 355\,396,\ 55\,597,\ 3\,752, 0, 0]\ ,$$

so that the largest number of breaks found was 5. Also as in Section VII, of interest is the distribution of the locations of the breaks; the number of elections that had a break in any of the five possible places from 1 to 7 was

$$[1\,165\,531,\ 828\,992,\ 605\,510,\ 522\,197,\ 604\,432,\ 826\,967,\ 1\,167\,239]\ .$$

So breaks may occur anywhere; and a substantial fraction

$$1\,165\,531/5\,154\,026 = 22.3\%$$

occur annoyingly between the candidates placed first and second, where they are sure to occasion controversy.

Of the elections studied $5\,075,863$ both yield a single rank order and, when the winning candidate is dropped, have the resulting election also yield a single rank order. In none of these did the candidate who originally placed third precede the candidate who originally placed second.

It follows from this observation that in our sample the candidate who originally placed third cannot become placed first, and that the candidate originally placed second cannot become placed last. The new position of the candidate who had placed third, minus the new position of the candidate who had placed second, runs from 1 to a maximum of 6; and the distribution of elections over those values runs

$$[5\,948\,981,\ 119\,441,\ 51\,222,\ 32\,829,\ 28\,102]\ .$$

So while the candidate who had placed third must continue to be placed behind the candidate who had placed second, he may appear anywhere behind.

Also of the 6 008 000 elections, $5\,075,863$ both yield a single rank order and, when the winning candidate is dropped by overwriting the victory matrix to make that candidate a Condorcet loser, has the resulting election also yield a single rank order. In none of these did the candidate who originally placed third precede the candidate who originally placed second.

It follows from this observation that the candidate who originally placed third cannot become placed first, and that the candidate originally placed second cannot become placed last. The new position of the candidate who had placed third, minus the new position of the candidate who had placed second, runs from 1 to a maximum of 6; and the distribution of elections over those values runs

$$[4\,927\,767,\ 75\,154,\ 28\,558,\ 17\,338,\ 14\,303,\ 12\,743]\ .$$

In our sample of elections, $5\,075,863$ had both a unique rank order and, when the winning candidate was dropped, that election too had a unique rank order. The following table $B^{(8)}_{jk}$ indicates, for the candidate who placed in position j in the original election, the position k in the second election where that same candidate placed; j runs 1 to 8, and k from 1 to 7.

$$B^{(8)} = \begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 4939364 & 108829 & 18014 & 5917 & 2620 & 1119 & 0 \\ 0 & 4808710 & 163933 & 43885 & 23194 & 18096 & 18045 \\ 34287 & 65225 & 4750997 & 149352 & 38940 & 20406 & 16656 \\ 25209 & 34176 & 82586 & 4728982 & 141905 & 39000 & 24005 \\ 26022 & 24861 & 32421 & 103190 & 4707090 & 134226 & 48053 \\ 27723 & 20278 & 18478 & 32388 & 132025 & 4699521 & 145450 \\ 23258 & 13784 & 9434 & 12149 & 30089 & 163495 & 4823654 \end{bmatrix}$$

XI. NINE TO EIGHTEEN CANDIDATES

For N from 9 to 18, numerical explorations show that when the N-candidate rank order is unique, and so is the rank order when the winner is dropped, that for some election the candidate in 2$^{\text{nd}}$ through N^{th} place in the original election will occupy any of the places 1$^{\text{st}}$ through $(N-1)^{\text{th}}$ in the following election—except that the candidate who placed second cannot become last, and the candidate who placed third cannot become first. These results are consistent with the conjecture the candidate who placed second always in the following election coming to place ahead of the candidate who placed third; a conjecture which was true in every election looked at, but which is of course not proved true by such tests.

As an example, over a sample of 200 000 elections with N-cycles for $N = 15$, I found 126 804 where the rank orders for the full election was unique, and the election when the winning candidate was dropped; and the various surviving candidates had the following distribution $B^{(15)}$ over their place in the second election; the only zeros are the two bolded ones. I would report the results for $N = 18$ instead, but that matrix would spill over the margins.

$$\begin{bmatrix} \cdots\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 125284 & 1289 & 161 & 29 & 18 & 3 & 1 & 6 & 2 & 4 & 3 & 1 & 3 & \mathbf{0} \\ \mathbf{0} & 124420 & 1763 & 269 & 75 & 31 & 19 & 18 & 13 & 19 & 25 & 32 & 51 & 69 \\ 126 & 305 & 124143 & 1721 & 259 & 66 & 45 & 20 & 22 & 17 & 12 & 20 & 24 & 24 \\ 79 & 109 & 309 & 124136 & 1707 & 276 & 63 & 26 & 18 & 16 & 15 & 11 & 16 & 23 \\ 59 & 68 & 106 & 349 & 124124 & 1666 & 246 & 63 & 37 & 24 & 24 & 14 & 12 & 12 \\ 39 & 49 & 53 & 97 & 374 & 124099 & 1655 & 243 & 62 & 35 & 27 & 31 & 21 & 19 \\ 64 & 50 & 44 & 42 & 109 & 432 & 124010 & 1608 & 226 & 76 & 46 & 37 & 32 & 28 \\ 75 & 42 & 29 & 36 & 39 & 112 & 505 & 123876 & 1574 & 257 & 110 & 56 & 45 & 48 \\ 104 & 62 & 41 & 24 & 29 & 42 & 147 & 637 & 123642 & 1510 & 249 & 157 & 91 & 69 \\ 117 & 76 & 42 & 30 & 27 & 29 & 58 & 186 & 844 & 123281 & 1463 & 298 & 197 & 156 \\ 152 & 88 & 39 & 24 & 20 & 18 & 28 & 72 & 236 & 1166 & 122817 & 1438 & 429 & 277 \\ 198 & 91 & 26 & 22 & 9 & 19 & 21 & 30 & 83 & 299 & 1591 & 122170 & 1542 & 703 \\ 242 & 81 & 29 & 16 & 4 & 8 & 4 & 16 & 36 & 82 & 336 & 2206 & 121685 & 2059 \\ 265 & 74 & 19 & 9 & 10 & 3 & 2 & 3 & 9 & 18 & 86 & 333 & 2656 & 123317 \end{bmatrix}$$

As one can see, for a large number of candidates N by far the most common occurrence when the candidate placed first drops out is for the candidate who had placed in position k to move up to position $k-1$. Except for it being impossible for the candidate who had been ranked third to precede the candidate who had been ranked second, which accounts for the two bolded zeros, any candidate can come to place anywhere in the new rank order, from first to last; though some shifts are observedly less probable than others.

XII. INDETERMINACY IN THE BEATPATH RANK ORDER

It seems that in the common case that while Beatpath can support multiple rank orders, it always has a unique winner and a unique loser. If so, then the most indeterminate an election can be is to have one candidate placed first, one placed last, and all the remaining candidates tied for second. A victory matrix V that achieves that result for given N can be constructed by filling the above-diagonal elements as follows.

Fill the row elements $V_{1,k}$ to $V_{1,N-1}$ left to right with consecutive decreasing integers, starting with $N(N-1)/2$. Continue to fill the column elements $V_{N-1,N}$ to $V_{1,N}$ bottom to top with decreasing integers. Fill the remaining above-diagonal elements with decreasing integers, first by column left to right, and then by row top to bottom. The above-diagonal elements are presently the positive integers 1 to $N(N-1)/2$; change the sign of the one corner element $V_{1,N}$ to make it negative. Now complete V to render the matrix antisymmetric. This victory matrix will have candidate 1 win, candidate N lose, and have the rest of the candidates tied in-between. (In fact the way one orders the integers in the row and column, so long as the element $(N^2-5N+8)/2$ remains in the upper-right corner, doesn't matter; and neither does the order one fills the triangle with the smaller integers.) For example for $N=6$ the victory matrix is

$$V = \begin{bmatrix} 0 & 15 & 14 & 13 & 12 & -7 \\ -15 & 0 & 6 & 5 & 4 & 8 \\ -14 & -6 & 0 & 3 & 2 & 9 \\ -13 & -5 & -3 & 0 & 1 & 10 \\ -12 & -4 & -2 & -1 & 0 & 11 \\ 7 & -8 & -9 & -10 & -11 & 0 \end{bmatrix},$$

which gives for the matrix giving the strength of the strongest paths just

$$Q = \begin{bmatrix} 0 & 15 & 14 & 13 & 12 & 11 \\ 7 & 0 & 7 & 7 & 7 & 8 \\ 7 & 7 & 0 & 7 & 7 & 9 \\ 7 & 7 & 7 & 0 & 7 & 10 \\ 7 & 7 & 7 & 7 & 0 & 11 \\ 7 & 7 & 7 & 7 & 7 & 0 \end{bmatrix},$$

and the partial order

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This construction shows how to construct a common-case victory matrix where the outcome under Beatpath is consistent with $(N-2)!$ rank orders. If it is true that Beatpath in the common case gives a unique winner and a unique loser in every election, then this is the largest number achievable.

XIII. DISCUSSION

Of what practical significance are the results in this paper for running mass popular elections using ranked-choice ballots?

Little; for any difference in the winner of a multi-candidate election under Ranked Pairs or under Beatpath will require the four (or more) leading candidates to be in a cycle of preferences (technically, for the Smith set of the election to feature 4 or more candidates). That will be a very rare form of election; so rare that choosing the best possible method to resolve the cycle and to choose a winner is of vastly less practical importance than correctly deciding garden-variety, three-candidate elections, even those few that might feature a cycle.

The subject having been raised, however: Ranked Pairs is better than Beatpath in this minor respect. It is plainly a useful property to have a guarantee that if the winning candidate drops, the order of the rest of the candidates cannot change, to ensure promoting the second-place candidate to first; and to have the parallel guarantee that if the losing candidate drops, the order of the rest of the candidates cannot change, ensuring we have the same winner as before.

And those properties Ranked Pairs has. Our examination of what can happen under Ranked Pairs and Beatpath in the case of 4 candidates, when any one of the candidates drops, did not reveal any measurable cost to using Ranked Pairs over Beatpath, so these properties can be had for essentially nothing.

Indeed, keeping these properties avoids ever having to explain why candidate A was elected instead of candidate B, who came second in the rank order, in the case when more voters preferred candidate B to A, than preferred A to B. While any four-cycle would be a rare event, the political penalty for having that event occur under Beatpath might be dire: it could call into question the legitimacy of an entire system of elections, if not indeed become a matter for revolution. Imagine the furor if in the United States we ever elected Donald Trump president over Joe Biden, or Biden over Trump, in a four-candidate election when a majority of the electorate indisputably and on record preferred his rival.

Not that Ranked Pairs cures all furor. Any election system that declares a rank order of the candidates in an election where the Smith set numbers 3 or more must declare some candidate to be higher in the rank order than another candidate who beat him, in the sense that a majority of the voters preferred the latter candidate to the former. But there are two significant advantages to using Ranked Pairs.

First, such a reversal does not affect candidates adjacent in the rank order; if I want to move above some candidate I beat, there is always some candidate between us in the rank order who beat me, in the sense that a majority of voters preferred him to me; and if I am to claim to be the proper winner I must also explain why I should place above that candidate. That will not be easy; the nature of Ranked Pairs is that if y beats x, and y trails x in the rank order, then for all the intervening candidates $a_1, a_2, \cdots a_n$ in the rank order, that not only does y beat a_1, and a_1 beat a_2, down to a_n beating x, but the margin of each of these contests is necessarily greater than the margin by which y beat x; a fact which undermines the credibility of the claim that y should place in the rank order above x.

Last, one must win a battle for the hearts and minds of voters either to see adopted or to maintain as adopted an election system that uses either Ranked Pairs or Beatpath. It is of course possible to define Beatpath only to declare a winner, and so to make no statement about the relative merits of the other candidates; but a system incapable of weighing those relative merits forfeits a great deal of popular legitimacy (in my view) to any method such as Ranked Pairs that can.

And if Beatpath is used to determine a rank order, it is not helpful, to win the hearts and minds of voters, to have to explain that if the winning candidate drops, that the hitherto losing candidate can come to be placed anywhere, including first; or that the candidate hitherto ranked second can come to be placed anywhere down to second-to-last.

It is true he cannot come to be placed last, but even that feature seems to be undercut; if it proves true, as our computer investigations suggest, that the candidate hitherto ranked third must always follow in the new rank order the candidate hitherto ranked second rank order, then the third-ranked candidate can never come to win. So whenever the candidate ranked second comes to be placed second-to-last, one must not only suffer the election of a candidate hitherto ranked at best as fourth, but the candidates hitherto ranked as second and third are both at the bottom of the rank order, respectively ranked last and second-to-last.

There is a truism among air-transport companies that if a passenger pulls down the tray in front of his seat and finds a coffee ring, he judges the company doesn't maintain its engines properly. People assess the worth of things too complicated for them to understand fully by the parts of them they think they do understand. I for one would not wish to persuade voters to adopt an election system by a campaign in which I would have to explain or excuse outcomes that are unimpeachably possible and strike voters as crazy. Particularly if my opponents could correctly point out I could have tried to pass a system like Ranked Pairs that was at least no worse in every other respect, and often better, and that avoided the apparently crazy outcomes.

There is another truism that in politics, if you are excusing, you are losing.

Appendix A: Proofs concerning Ranked Pairs

In this appendix we provide for the reader's convenience proofs of pertinent results concerning Ranked Pairs that are almost folk-theorems, in that the easiest way to prove them isn't immediately obvious, but once found it is simple enough that it is often omitted from academic publications. The proofs in the separate sections can be read independently, except that the result that Ranked Pairs has inversion symmetry (A 2) we use to prove it is also leader drop-steady (A 4).

1. Proofs that Ranked Pairs is both Smith and ISDA

Recall that Ranked Pairs works from the antisymmetric victory matrix V, where V_{jk} is the number of ballots on which candidate j is placed above candidate k, minus the number on which k is placed above j. Ranked Pairs collects all the non-diagonal elements V_{jk} that are positive, and puts the corresponding pairs (j, k) in a list sorted in order of decreasing size of V_{jk}. Ranked Pairs then builds an accepted list of pairs, initially empty, such that an accepted pair (j, k) means that candidate j will precede candidate k in the final rank order of the candidates. Each pair in the sorted list is considered in turn to be added to the accepted list. A pair is added unless it would create in the accepted list some cycle or loop of preferences (for example that j would precede k, and k would precede l; but the new pair would have that l precede j), whereupon that pair would be rejected and the next pair on the sorted list considered. When all the pairs on the sorted list have been considered, the pairs on the accepted list will be consistent with only one (total) order of the candidates, which becomes the rank order produced by Ranked Pairs.

The Smith set is the smallest nonempty set of candidates S such for every candidate j in S and every candidate k not in S, we have $V_{jk} > 0$.

Claim. In Ranked Pairs no candidate in not-S can ever precede in the rank order any candidate in S.

Proof. If k is in not-S and j is, clearly the algorithm can never impose that k precede j by means of accepting the pair (k, j), because $V_{kj} < 0$ and so the pair (k, j) could never even appear on the sorted list. Therefore that k must precede j would have to be imposed indirectly by rejecting the pair (j, k), which does appear on the sorted list. But that rejection could happen only if the acceptance of the pair (j, k) would create a cycle or loop of preferences in the accepted list. Such a loop would have to include a candidate in S, at least one transition from a candidate in S to a candidate not in S (or the loop could not include k), and then at least one transition from a candidate not in S back to a candidate in S (or the loop could not include j).

But no pair from not-S to S will ever on the sorted list to begin with, because for any such pair (b, a) we must have $V_{ba} < 0$. Therefore every candidate in S precedes any candidate in not-S. •

Therefore Ranked Pairs is Smith [5]. A similar argument shows Ranked Pairs is independent of Smith-dominated alternatives. For we have the following.

Claim. The rank order of the candidates in the Smith set is independent of the presence on the ballot of any candidates not in the Smith set.

Proof. Suppose when only the candidates in the Smith set are present on the ballot, that candidate a_1 precedes in the rank order candidate a_2. Now add to the ballot all the candidates not in the Smith set. The pair (a_1, a_2) remains on the list of sorted pairs; to change the rank order so that a_2 instead precedes a_1, that pair must be rejected. That is possible only if there is some candidate b in not-S so that if the pair (a_1, a_2) is accepted, then in the accepted list a cycle or loop or preferences would be created. Such a loop would have to run from a_1 to a_2, and include a transition from a candidate in S to a candidate in not-S (or the loop could not include b), and also include a transition from not-S to S (or the loop could not enter S to close the loop on a_1).

But as we have seen no pair from not-S to S could ever appear on the sorted list to begin with; and therefore no such loop is possible. Therefore in the presence of the candidates in not-S, candidate a_1 must continue to precede a_2. Therefore the order of the candidates in S is independent of the presence of any candidates in not-S. •

2. Proof that Ranked Pairs has Inversion symmetry

An election method whose sole rank order depends only upon the victory matrix is said to have inversion symmetry if, when the victory matrix changes sign, the rank order of the candidates reverses; this if V provides the rank order $ABCD$, then $-V$ provides the rank order $DCBA$.

In the common case, the sole effect of changing the sign of V is to replace each pair of candidates (a, b) on the sorted list with its reversal, (b, a). As the Ranked Pair algorithm proceeds, pairs fail to be added to the accepted list only if they would create a loop or cycle of preferences; but if (x, y) would create a cycle in the original acceptance list, (y, x) must create a cycle in the acceptance list with pairs on it all reversed. So a pair (x, y) is accepted under V if and only if the pair (y, x) is accepted under $-V$. Therefore when V changes sign, the only effect is that any constraint that x precede y in the rank order is replaced with the constraint that y precede x; and therefore the rank order must reverse. •

3. Proof that Ranked Pairs is trailer drop-steady

An election method whose sole rank order depends only upon the victory matrix is said to be trailer drop-steady if, when the candidate last on the rank order is dropped from the election, and so when the corresponding row and column for that candidate in the victory matrix are deleted, the rank order for the remaining candidates is unchanged.

Under Ranked Pairs, a candidate x can become ranked last if and only if either there are no pairs on the sorted list of the form (x, a), where a is any other candidate; or if there are, then in each case the acceptance list has evolved so that the pair (x, a) cannot be accepted. Therefore the rank order would not change had we deleted all the pairs (x, a), if any, from the sorted list from the beginning. That being so, the only effect of the pairs on that list of the form (a, x) is to ensure that x is ranked behind all the candidates a; those pairs have no effect on the order of the candidates a among themselves, because without any pairs of the form (x, a) there are no loops that could form in the acceptance list that involve x. Therefore the order of the candidates a would be the same if all the pairs that involve x had been deleted from the sorted list; which is the effect of deleting the row and column in V that affect x. •

4. Proof that Ranked Pairs is leader-drop-steady

An election method whose sole rank order depends only upon the victory matrix is said to be leader drop-steady if, when the candidate first on the rank order is dropped from the election, and so when the corresponding row and column for that candidate in the victory matrix are deleted, the rank order for the remaining candidates is unchanged.

Any example of Ranked Pairs failing to be leader drop-steady would, using its inversion symmetry (see Appendix A 2), become an example of Ranked Pairs failing to be trailer drop-steady; for example, if $ABCD$ when A dropped produced not BCD but say CDB, then on changing the sign of V we would have $DCBA$ when A dropped producing not DCB but BDC. Because Ranked Pairs is trailer drop-steady (A 3), it is therefore leader drop-steady as well. •

Appendix B: Variant proof of Schulze's theorem concerning Beatpath

In this appendix we present the alternative and derivative proof, mentioned in Section IV, of the limited form of a theorem proved by M. Schulze [3]. In this appendix, for one candidate to "beat" another means that the one must precede the other in the partial order computed by Beatpath.

Theorem. Suppose the number of candidates N is 3 or more. Suppose that under Beatpath

(a) Candidate a beats all other candidates;

(b) Candidate b beats or ties all other candidates but a.

Then if candidate a drops, in the smaller election candidate b must beat at least one candidate.

We will need the following lemma.

Lemma. For any election we claim that we can construct a set of $N(N-1)/2$ paths, one running from candidate j to candidate k for any such pair of candidates, such that the strength of each path equals the greatest strength for any possible path running from j and k; and such that the paths nest, in the following sense: in the string of consecutive nodes in the path running from j to k, any consecutive substring is also one of the $N(N-1)/2$ paths in the set.

Proof. To construct such a set, denote the different candidates by distinct letters. Note that in any path we can excise any stretch of candidates that appear between a pair of candidates that are the same, without diminishing the strength of the resulting path, e.g., we can have all the following three kinds of contractions:

$$\mathbf{j}cde\mathbf{j}fk \to jfk$$
$$j\mathbf{c}edf\mathbf{c}gk \to jcgk$$
$$jcde\mathbf{k}fg\mathbf{k} \to jcdek$$

Therefore to find all the strongest path from j to k, we need examine only a finite set. We can thus find all the paths from j to k of (equal) greatest strength; of these, select a path with the fewest candidates between j and k. Should more than one path tie both for greatest strength and for having the fewest candidates, the paths have the same length; sort the paths into alphabetical order, and choose the path at the head of the list. Thus if $jdek$, $jdfk$, and $jchk$ so tie, choose the path $jchk$.

The collection of single paths, one for each j and k, is our desired set. To show this, we test for nesting all the two-element, then three-element, then four-element paths, and so on; and within each group, test that the paths sorted alphabetically.

Consider a two-candidate path ef. Clearly that path cannot reoccur among any other of the two-candidate paths. Can the candidates e and f occur separated by some other candidate or candidates in some path of greater length, as in $jc\mathbf{e}gh\mathbf{f}ik$? Clearly not, because ef has the greatest strength of any path between e and f, so we could replace the stretch $\mathbf{e}gh\mathbf{f}$ with a plain ef without reducing the strength of the path. But that would create a strongest path between j and k that has fewer candidates than the one already chosen, whose number of candidates was already minimal. So all the two-elements paths nest properly among the paths of greater length.

That established, what of a three-element path, say cgf? Clearly there is only one 3-element path from c to f on our list; a second not on our list of equal strength might exist, say chf; but of these we choose only one because only one can be at the head of list sorted by alphabet. Can the candidates c and f occur separated by some other candidate or candidates in some path of greater length? Not if the number of candidates between number 2 or more, as in $jcdefhk$, or we could get a path of equivalent or greater strength by replacing the stretch $cdef$ with chf; but that would create a strongest path between j and k that has fewer candidates than the one chosen, whose number of candidates was already minimal. Or if the number of candidates were 1, as in $jchfk$; then while that path and $jcgfk$ might be of equal strength and equal length, we would not have put $jchfk$ on our list because of the rule that we would have selected the path closest to the head of a list sorted into alphabetical order.

All the paths so chosen therefore nest, proving the Lemma. •

The conceptual scheme above is not at all efficient computationally, of course. As observed by M. Schulze [3], the Floyd-Warshall algorithm, when used not only to find the strength of a strongest path from j to k, but one representative of the set of strongest paths, automatically produces a set of strongest paths that nest, in a time that is $O(N^3)$, where N is the number of candidates. That is a stronger result, but it takes more work to prove.

For brevity, refer to the particular path from one candidate to another, whose strength equals the maximum strength of any path from the one to the other, as the Path (note the capital) from the one to the other, if it is the path in the set of nested paths constructed by the Lemma.

We can now prove the theorem by establishing two successive claims. Call the election with all N candidates the *old* election and the election with $N-1$ candidates the *new* election.

Out of the candidates in the old election choose two, which we will call α and β. Any candidate not either α or β we will indicate generically as a candidate x. Suppose every Path that begins with β contains α, either at its end or between β and its end. We will then call α a *companion* of β.

Because all Paths, between whatever pair of candidates, nest, if a Path begins with β, then α is the candidate immediately after β. For if α is a companion of β, and a path of the form $\beta\, x_1\, x_2\, \alpha\, x_3 \cdots$ occurred, then because all Paths nest, both $\beta\, x_1$ and $\beta\, x_1\, x_2$ would be Paths; but since neither contains α, we would contradict that α is a companion of β. In particular the simple two-element link $\beta\,\alpha$ must be a Path.

Now strengthen our assumptions about α: only will it be a companion to β, but it will be the winner of the old election. A candidate with both properties we shall denote by a. There must be a Path from a to β, and there are only two forms that Path might take: either there is no candidate x between a and β, so that the Path is the simple link $a\,\beta$; or there are some candidates x in between a and β, so that the Path is $a\,x' \cdots \beta$ for some initial candidate x' followed by some number, perhaps zero, of other candidates x.

Claim: the condition that in the Path from a to β that a candidate x' follow a, and the condition that β beats or ties x', are incompatible.

Proof. There must be a Path from a to β; it is either $a\,\beta$; or a Path with some intermediate number of candidates x in between, a Path such as $a\,x' \cdots \beta$. Since a beats β, the strength of the strongest path from a to β must exceed (>) the strength of the strongest path from β to a. In particular, the strength of any subsection of the strongest path from a to β must exceed (>) the strength of the strongest path from β to a. Hence the strength of the path $x' \cdots \beta$, which is a subsection of the path $a\,x' \cdots \beta$, exceeds the strength of the path $\beta\,a$, which we know to be the strongest path from β to a. We may write this comparison of the strength of two paths as the inequality $S(x' \cdots \beta) > S(\beta\, a)$.

We also know from nesting that if x' follows a in a Path, that $a\, x'$ is the Path from a to x'; and we know that $\beta\, a$ is the Path from β to a; and so $\beta\, a\, x'$ is the Path from β to x'. If β were to at least tie x', the strength of the path $\beta\, a\, x'$ would at least tie the strength of the path $x' \cdots \beta$. In particular a subsection of the path $\beta\, a\, x'$ would have a strength that would at least tie the strength of the path $x' \cdots b$; and so the strength of βa would at least tie $x' \cdots \beta$. Briefly, we would have $S(x' \cdots \beta) \leq S(\beta\, a)$.

The claim follows from the inequalities $S(x' \cdots \beta) > S(\beta\, a)$ and $S(x' \cdots \beta) \leq S(\beta\, a)$ being incompatible. •

Claim: if the Path from a to β is $a\,\beta$, then every candidate x beats β; and so β must be in last place.

Proof. The assumption that the Path $a\,\beta$ is the strongest path from a to β, and the facts that the Path $\beta\, a$ is the Path from β to a, and that a beats β, imply that the strength of the path $a\,\beta$ is > 0 and the strength of the path $\beta\, a$ is < 0. (Alternatively, that $V_{a\beta} > 0$, and since the victory matrix is antisymmetric, that $V_{\beta a} < 0$.) The Path from β to any candidate x leads with the sequence $\beta\, a \cdots x$; since the link βa has negative (< 0) strength, the strength of the Path from β to any x must be < 0. Now $\beta\, x$ is a path; since its strength must be $\leq$ the strength of the strongest path from β to x, the strength of the link $\beta\, x$ must be < 0.

If the strength of the path $\beta\, x$ is < 0, clearly the strength of the reverse path $x\,\beta$ is > 0, because the first has strength $V_{\beta x}$ and the latter strength $V_{x\beta}$, and V is antisymmetric. So we have found a path from x to β whose strength is positive; and therefore the strength of the strongest path from x to β must be positive; while the greatest strength of any return path $\beta\, a \cdots x$ is negative. Therefore x beats β. Since x was arbitrary, any candidate x beats β. Since a also beats β, candidate β must be in sole last place. •

If we combine the two claims, either the strongest path from a to β takes the form $a\,\beta$, when every candidate beats β; or the strongest path has the form $a\, x' \cdots \beta$ for some candidate x', when x' beats β. Whichever condition holds, if a is a companion of β and a is a sole winner, then at least one candidate (other than a) beats β, and so candidate β cannot be in sole second place, nor tied for second place.

Therefore every candidate that has a companion who is the winner of the election cannot be in sole second place, nor tied for second place.

We can now prove the main theorem by reaching a contradiction. Consider a hypothetical candidate q who is in second place or tied for second place, but when the candidate w in first place drops, candidate q falls to sole last place. For every candidate x, it will be necessary for w to appear in the strongest path from q to x, or there is no way to change from x beating q in the race without w, which x must because q is there in sole last place, and yet have q beating x in the race with w. Clearly w also appears in the strongest path from q to w. Therefore w would have to be a companion to q; but if he were, we reach, contrary to our assumption, that q cannot be cannot be in second place or tied for second place. Therefore q cannot exist. ■

Appendix C: For $N = 4$, 5, 6, and 7, victory matrices of interest under Beatpath when the winning candidate drops out.

We have claimed that when the winning candidate drops out of a race under Beatpath, that it is possible for any remaining candidate to come to occupy any place in the new rank order, with the exceptions that the candidate who had placed second cannot come to be placed last, and the candidate who had been ranked third cannot come to be ranked first. We here provide, for values of N from 4 to 7, examples of victory matrices where each remaining candidate comes to occupy each of the places we have found to be possible.

In the arrays below, the line of $N(N-1)/2$ integers to the left of the vertical bar are the above diagonal elements of the victory matrix, listed in the order of the column index increasing fastest; thus for $N = 1$ they are listed left to right in the order V_{12}, V_{13}, V_{14}, V_{23}, V_{24}, V_{34}. The matrices have been chosen so that under Beatpath the rank order for that victory matrix is unique and is the order $[1, 2, \cdots, N]$, so that candidate 1 is first in the rank order and candidate N is the last, with the rest of the candidates in order. For each matrix, listed to the right of the vertical bar is the rank order that results when candidate 1, who had placed first, drops out. Thus the second line of the array for $N = 4$ should be read as, the victory matrix

$$\begin{pmatrix} 0 & 4 & -3 & 6 \\ -4 & 0 & 5 & -1 \\ 3 & -5 & 0 & -2 \\ -5 & 1 & 1 & 0 \end{pmatrix}$$

has rank order [1, 2, 3, 4]; when candidate 1 drops, the new rank order is [4, **2**, 3]. The bolded digit shows that this rank order has been chosen to put candidate 2 in second place after candidate 1 drops. Some matrices are used in multiple places in an array.

$$N = 4$$

$$\left[\begin{array}{rrrrrr|ccc}
3 & 6 & -1 & 5 & -2 & 4 & \mathbf{2} & 3 & 4 \\
4 & -3 & 6 & 5 & -1 & -2 & 4 & \mathbf{2} & 3 \\
. & . & . & . & . & . & . & . & \cancel{2} \\
. & . & . & . & . & . & \cancel{3} & . & . \\
3 & 6 & -1 & 5 & -2 & 4 & 2 & \mathbf{3} & 4 \\
5 & -4 & 3 & 6 & 1 & -2 & 2 & 4 & \mathbf{3} \\
4 & -3 & 6 & 5 & -1 & -2 & \mathbf{4} & 2 & 3 \\
5 & -4 & 3 & 6 & 1 & -2 & 2 & \mathbf{4} & 3 \\
3 & 6 & -1 & 5 & -2 & 4 & 2 & 3 & \mathbf{4}
\end{array}\right]$$

$$N = 5$$

$$\left[\begin{array}{rrrrrrrrrr|cccc}
1 & 6 & -2 & -3 & -5 & 4 & 8 & 9 & -7 & 10 & \mathbf{2} & 3 & 4 & 5 \\
7 & -6 & 5 & -3 & 8 & -2 & 9 & -4 & 10 & -1 & 4 & \mathbf{2} & 3 & 5 \\
8 & -5 & 9 & 7 & 6 & -2 & -1 & -3 & -4 & 10 & 4 & 5 & \mathbf{2} & 3 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & \cancel{2} \\
. & . & . & . & . & . & . & . & . & . & \cancel{3} & . & . & . \\
1 & 6 & -2 & -3 & -5 & 4 & 8 & 9 & -7 & 10 & 2 & \mathbf{3} & 4 & 5 \\
7 & -6 & 5 & -3 & 8 & -2 & 9 & -4 & 10 & -1 & 4 & 2 & \mathbf{3} & 5 \\
10 & -6 & -2 & 4 & 8 & 7 & -5 & 3 & -1 & 9 & 2 & 4 & 5 & \mathbf{3} \\
7 & -6 & 5 & -3 & 8 & -2 & 9 & -4 & 10 & -1 & \mathbf{4} & 2 & 3 & 5 \\
10 & -6 & -2 & 4 & 8 & 7 & -5 & 3 & -1 & 9 & 2 & \mathbf{4} & 5 & 3 \\
1 & 6 & -2 & -3 & -5 & 4 & 8 & 9 & -7 & 10 & 2 & 3 & \mathbf{4} & 5 \\
7 & 1 & -5 & -3 & 6 & -2 & 10 & 8 & 9 & -4 & 2 & 3 & 5 & \mathbf{4} \\
-5 & 6 & -4 & 7 & 9 & -8 & -3 & 10 & 1 & -2 & \mathbf{5} & 2 & 3 & 4 \\
7 & -3 & -5 & 6 & 8 & -2 & 9 & 10 & -4 & -1 & 2 & \mathbf{5} & 3 & 4 \\
7 & 1 & -5 & -3 & 6 & -2 & 10 & 8 & 9 & -4 & 2 & 3 & \mathbf{5} & 4 \\
1 & 6 & -2 & -3 & -5 & 4 & 8 & 9 & -7 & 10 & 2 & 3 & 4 & \mathbf{5}
\end{array}\right]$$

$N = 6$

$$
\left[\begin{array}{rrrrrrrrrrrrrrr|ccccc}
13 & -3 & -4 & 1 & -6 & 10 & -9 & 14 & 11 & 12 & -7 & 8 & 15 & -2 & 5 & \mathbf{2} & 3 & 4 & 5 & 6 \\
13 & -10 & 14 & -7 & 15 & 11 & -4 & 6 & -5 & -8 & -2 & 1 & 12 & -3 & 9 & 4 & \mathbf{2} & 5 & 6 & 3 \\
14 & -12 & 11 & 13 & -5 & 15 & 3 & -7 & 9 & -10 & 1 & 8 & -2 & -4 & -6 & 4 & 5 & \mathbf{2} & 3 & 6 \\
15 & -11 & 9 & 13 & 10 & 14 & -8 & -7 & -1 & -4 & 3 & -5 & 12 & -2 & -6 & 6 & 4 & 5 & \mathbf{2} & 3 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & \cancel{2} \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & \cancel{3} & . & . & . & . \\
13 & -3 & -4 & 1 & -6 & 10 & -9 & 14 & 11 & 12 & -7 & 8 & 15 & -2 & 5 & 2 & \mathbf{3} & 4 & 5 & 6 \\
14 & 1 & -8 & -10 & 7 & 15 & -4 & -2 & 12 & 13 & 9 & -5 & 11 & -3 & -6 & 2 & 6 & \mathbf{3} & 4 & 5 \\
14 & -12 & 11 & 13 & -5 & 15 & 3 & -7 & 9 & -10 & 1 & 8 & -2 & -4 & -6 & 4 & 5 & 2 & \mathbf{3} & 6 \\
13 & -12 & -4 & -9 & 11 & 14 & 10 & -1 & 7 & -5 & 3 & -8 & 15 & -2 & 6 & 2 & 4 & 5 & 6 & \mathbf{3} \\
13 & -10 & 14 & -7 & 15 & 11 & -4 & 6 & -5 & -8 & -2 & 1 & 12 & -3 & 9 & \mathbf{4} & 2 & 5 & 6 & 3 \\
13 & -12 & -4 & -9 & 11 & 14 & 10 & -1 & 7 & -5 & 3 & -8 & 15 & -2 & 6 & 2 & \mathbf{4} & 5 & 6 & 3 \\
13 & -3 & -4 & 1 & -6 & 10 & -9 & 14 & 11 & 12 & -7 & 8 & 15 & -2 & 5 & 2 & 3 & \mathbf{4} & 5 & 6 \\
14 & 1 & -8 & -10 & 7 & 15 & -4 & -2 & 12 & 13 & 9 & -5 & 11 & -3 & -6 & 2 & 6 & 3 & \mathbf{4} & 5 \\
12 & 10 & -8 & -6 & 11 & 15 & 1 & 5 & -7 & 9 & 14 & -3 & -2 & 4 & 13 & 2 & 3 & 5 & 6 & \mathbf{4} \\
9 & -8 & -2 & 12 & -6 & 13 & -11 & -4 & 14 & 15 & -1 & 7 & 3 & -5 & 10 & \mathbf{5} & 2 & 3 & 4 & 6 \\
11 & 6 & -8 & 13 & -5 & 9 & 3 & 12 & -1 & 15 & -7 & 10 & -4 & -2 & 14 & 2 & \mathbf{5} & 3 & 6 & 4 \\
13 & -12 & -4 & -9 & 11 & 14 & 10 & -1 & 7 & -5 & 3 & -8 & 15 & -2 & 6 & 2 & 4 & \mathbf{5} & 6 & 3 \\
13 & -3 & -4 & 1 & -6 & 10 & -9 & 14 & 11 & 12 & -7 & 8 & 15 & -2 & 5 & 2 & 3 & 4 & \mathbf{5} & 6 \\
12 & 9 & -2 & -7 & 14 & 13 & 11 & 3 & -5 & 8 & 6 & -1 & 15 & 10 & 4 & 2 & 3 & 4 & 6 & \mathbf{5} \\
14 & -12 & -7 & -1 & 11 & 15 & 3 & -8 & 2 & 9 & 10 & -5 & 13 & -6 & 4 & \mathbf{6} & 2 & 3 & 4 & 5 \\
14 & 1 & -8 & -10 & 7 & 15 & -4 & -2 & 12 & 13 & 9 & -5 & 11 & -3 & -6 & 2 & \mathbf{6} & 3 & 4 & 5 \\
13 & 8 & -10 & -5 & 3 & 15 & 9 & -2 & 11 & 14 & 7 & 12 & -6 & -1 & -4 & 2 & 3 & \mathbf{6} & 5 & 4 \\
13 & -12 & -4 & -9 & 11 & 14 & 10 & -1 & 7 & -5 & 3 & -8 & 15 & -2 & 6 & 2 & 4 & 5 & \mathbf{6} & 3 \\
13 & -3 & -4 & 1 & -6 & 10 & -9 & 14 & 11 & 12 & -7 & 8 & 15 & -2 & 5 & 2 & 3 & 4 & 5 & \mathbf{6}
\end{array}\right]
$$

$$N = 7$$

$$\left[\begin{array}{rrrrrrrrrrrrrrrrrrrrr|cccccc}
17 & -14 & 8 & -13 & -7 & 10 & 16 & -11 & -9 & 4 & -2 & 19 & -18 & 5 & 20 & 21 & 15 & 6 & 1 & -3 & 12 & \mathbf{2} & 3 & 4 & 5 & 6 & 7 \\
17 & -4 & -14 & 16 & 13 & 11 & 19 & 7 & -12 & 3 & 18 & 15 & 6 & 20 & -5 & 1 & -10 & -8 & 21 & -9 & 2 & 5 & \mathbf{2} & 7 & 3 & 6 & 4 \\
16 & -2 & -15 & 17 & 9 & 4 & 20 & -11 & -7 & -10 & 13 & 19 & 6 & -12 & 3 & 8 & -14 & -5 & 21 & -1 & 18 & 5 & 6 & \mathbf{2} & 3 & 4 & 7 \\
14 & -13 & 18 & -4 & 9 & 8 & 20 & 5 & 3 & -11 & -2 & -6 & -12 & -10 & 19 & 17 & -1 & 16 & 21 & 15 & -7 & 4 & 5 & 6 & \mathbf{2} & 3 & 7 \\
18 & -14 & 21 & -11 & 15 & 7 & 19 & -9 & -3 & -8 & -2 & -12 & 4 & 5 & -1 & 16 & 6 & -13 & 20 & 10 & 17 & 4 & 5 & 6 & 7 & \mathbf{2} & 3 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 2 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 3 & . & . & . & . & . \\
17 & -14 & 8 & -13 & -7 & 10 & 16 & -11 & -9 & 4 & -2 & 19 & -18 & 5 & 20 & 21 & 15 & 6 & 1 & -3 & 12 & 2 & \mathbf{3} & 4 & 5 & 6 & 7 \\
17 & -9 & 14 & 10 & 4 & -5 & 16 & 15 & -2 & 20 & 11 & -7 & 19 & 8 & 12 & 1 & -6 & 21 & 18 & -3 & 13 & 2 & 4 & \mathbf{3} & 5 & 6 & 7 \\
17 & -4 & -14 & 16 & 13 & 11 & 19 & 7 & -12 & 3 & 18 & 15 & 6 & 20 & -5 & 1 & -10 & -8 & 21 & -9 & 2 & 5 & 2 & 7 & \mathbf{3} & 6 & 4 \\
15 & -13 & 14 & -3 & 19 & 5 & 18 & 9 & 17 & -11 & 6 & -1 & 7 & 8 & 12 & 16 & 21 & -4 & 20 & -2 & 10 & 4 & 2 & 5 & 6 & \mathbf{3} & 7 \\
20 & -15 & -12 & 5 & 3 & 7 & 17 & 13 & 4 & 14 & -11 & -6 & 9 & -2 & 8 & 21 & -16 & 1 & 18 & -10 & 19 & 2 & 4 & 5 & 6 & 7 & \mathbf{3} \\
15 & -13 & 14 & -3 & 19 & 5 & 18 & 9 & 17 & -11 & 6 & -1 & 7 & 8 & 12 & 16 & 21 & -4 & 20 & -2 & 10 & \mathbf{4} & 2 & 5 & 6 & 3 & 7 \\
21 & -12 & 16 & 7 & 14 & -4 & 18 & -5 & 20 & 15 & 6 & 3 & -11 & 9 & 8 & 17 & -10 & -1 & 19 & -2 & 13 & 2 & \mathbf{4} & 5 & 6 & 3 & 7 \\
17 & -14 & 8 & -13 & -7 & 10 & 16 & -11 & -9 & 4 & -2 & 19 & -18 & 5 & 20 & 21 & 15 & 6 & 1 & -3 & 12 & 2 & 3 & \mathbf{4} & 5 & 6 & 7 \\
18 & -6 & 7 & -11 & 19 & -9 & 14 & 17 & -3 & -8 & 15 & 13 & 16 & -1 & 21 & 20 & -5 & 2 & 4 & 10 & 12 & 6 & 2 & 3 & \mathbf{4} & 5 & 7 \\
17 & 9 & -16 & 19 & 14 & -10 & 18 & 20 & -2 & 11 & 5 & 21 & 6 & -12 & 1 & 7 & 4 & 15 & 13 & -8 & 3 & 2 & 5 & 6 & 3 & \mathbf{4} & 7 \\
17 & -4 & -14 & 16 & 13 & 11 & 19 & 7 & -12 & 3 & 18 & 15 & 6 & 20 & -5 & 1 & -10 & -8 & 21 & -9 & 2 & 5 & 2 & 7 & 3 & 6 & \mathbf{4} \\
17 & -4 & -14 & 16 & 13 & 11 & 19 & 7 & -12 & 3 & 18 & 15 & 6 & 20 & -5 & 1 & -10 & -8 & 21 & -9 & 2 & \mathbf{5} & 2 & 7 & 3 & 6 & 4 \\
17 & 9 & -16 & 19 & 14 & -10 & 18 & 20 & -2 & 11 & 5 & 21 & 6 & -12 & 1 & 7 & 4 & 15 & 13 & -8 & 3 & 2 & \mathbf{5} & 6 & 3 & 4 & 7 \\
15 & -13 & 14 & -3 & 19 & 5 & 18 & 9 & 17 & -11 & 6 & -1 & 7 & 8 & 12 & 16 & 21 & -4 & 20 & -2 & 10 & 4 & 2 & \mathbf{5} & 6 & 3 & 7 \\
17 & -14 & 8 & -13 & -7 & 10 & 16 & -11 & -9 & 4 & -2 & 19 & -18 & 5 & 20 & 21 & 15 & 6 & 1 & -3 & 12 & 2 & 3 & 4 & \mathbf{5} & 6 & 7 \\
16 & -5 & 4 & 13 & -14 & 2 & 19 & -18 & 11 & 7 & 8 & 20 & -1 & 3 & 21 & 15 & 12 & -10 & 17 & -6 & 9 & 2 & 3 & 4 & 7 & \mathbf{5} & 6 \\
17 & 6 & 18 & -10 & -9 & 12 & 15 & -11 & 20 & 14 & 19 & 21 & 5 & 4 & -1 & -3 & -2 & 13 & -7 & -8 & 16 & 2 & 3 & 4 & 6 & 7 & \mathbf{5} \\
18 & -6 & 7 & -11 & 19 & -9 & 14 & 17 & -3 & -8 & 15 & 13 & 16 & -1 & 21 & 20 & -5 & 2 & 4 & 10 & 12 & \mathbf{6} & 2 & 3 & 4 & 5 & 7 \\
16 & -2 & -15 & 17 & 9 & 4 & 20 & -11 & -7 & -10 & 13 & 19 & 6 & -12 & 3 & 8 & -14 & -5 & 21 & -1 & 18 & 5 & \mathbf{6} & 2 & 3 & 4 & 7 \\
17 & 9 & -16 & 19 & 14 & -10 & 18 & 20 & -2 & 11 & 5 & 21 & 6 & -12 & 1 & 7 & 4 & 15 & 13 & -8 & 3 & 2 & 5 & \mathbf{6} & 3 & 4 & 7 \\
15 & -13 & 14 & -3 & 19 & 5 & 18 & 9 & 17 & -11 & 6 & -1 & 7 & 8 & 12 & 16 & 21 & -4 & 20 & -2 & 10 & 4 & 2 & 5 & \mathbf{6} & 3 & 7 \\
17 & -14 & 8 & -13 & -7 & 10 & 16 & -11 & -9 & 4 & -2 & 19 & -18 & 5 & 20 & 21 & 15 & 6 & 1 & -3 & 12 & 2 & 3 & 4 & 5 & \mathbf{6} & 7 \\
16 & -5 & 4 & 13 & -14 & 2 & 19 & -18 & 11 & 7 & 8 & 20 & -1 & 3 & 21 & 15 & 12 & -10 & 17 & -6 & 9 & 2 & 3 & 4 & 7 & 5 & \mathbf{6} \\
12 & -11 & -5 & 6 & 19 & 20 & -10 & 9 & 7 & 14 & 2 & 16 & -15 & 18 & -8 & 17 & -13 & -1 & 21 & -3 & -4 & \mathbf{7} & 2 & 3 & 4 & 5 & 6 \\
16 & -7 & -6 & -14 & -9 & 20 & 1 & 19 & -4 & 8 & 17 & 18 & -15 & -12 & 3 & 21 & 13 & 10 & 2 & -11 & -5 & 2 & \mathbf{7} & 3 & 4 & 5 & 6 \\
17 & -4 & -14 & 16 & 13 & 11 & 19 & 7 & -12 & 3 & 18 & 15 & 6 & 20 & -5 & 1 & -10 & -8 & 21 & -9 & 2 & 5 & 2 & \mathbf{7} & 3 & 6 & 4 \\
16 & -5 & 4 & 13 & -14 & 2 & 19 & -18 & 11 & 7 & 8 & 20 & -1 & 3 & 21 & 15 & 12 & -10 & 17 & -6 & 9 & 2 & 3 & 4 & \mathbf{7} & 5 & 6 \\
-6 & 7 & 19 & 11 & -10 & 20 & 21 & -15 & -16 & 9 & 2 & 17 & -13 & -4 & 12 & 18 & 1 & 5 & 14 & -8 & -3 & 2 & 3 & 4 & 5 & \mathbf{7} & 6 \\
17 & -14 & 8 & -13 & -7 & 10 & 16 & -11 & -9 & 4 & -2 & 19 & -18 & 5 & 20 & 21 & 15 & 6 & 1 & -3 & 12 & 2 & 3 & 4 & 5 & 6 & \mathbf{7}
\end{array}\right]$$

[1] M. Schulze, *A new monotonic and clone-independent single-winner election method*, Voting Matters, issue 17, pp. 9–19, 2003; M. Schulze, *A new monotonic, clone-independent, reversal symmetric, and condorcet-consistent* [sic.] *single-winner election method,* Social Choice and Welfare 3, pp. 267–303, 2011; and M. Schulze, The Schulze Method of Voting, https:\\arxiv.org\ftp\arxiv\papers\1804\1804.02973.pdf.

[2] T.N. Tideman, *Independence of Clones as a Criterion for Voting Rules*, Social Choice and Welfare 4, pp. 185–206, 1987; T.M. Zavist and T. N. Tideman, *Complete Independence of Clones in the Ranked Pairs Rule*, Social Choice and Welfare 6, pp. 167–173, 1989.

[3] M. Schulze, *The Schulze Method of Voting,* https:\\arxiv.org\ftp\arxiv\papers\1804\1804.02973.pdf. The theorem and its proof appear in Section 4.21, pp. 283–285.

[4] J.W. Moon, *Topics on Tournaments*, Holt, Rinehart and Winston, New York, 1968, Library of Congress Catalog Card Number 68-13611 2676302, available at https:\\www.gutenberg.org\files\42833\42833-pdf.pdf/ For information on irreducibility, see Section 2, pp. 2–6.

[5] For a proof that Ranked Pairs is Smith see T.N. Tideman, *Collective Decisions and Voting: The Potential for Public*

Choice, Routledge, London, 2006 and 2016, p. 221, and Table 13.1 on p. 237; and see Appendix A 1. For proofs that Ranked Pairs is both trailer drop-steady and leader drop-steady, see Appendices A 3 and A 4.

[6] For a proof that Beatpath is Smith and ISDA see ref. [3], Section 4.8, pp. 228–231. In this reference independence of irrelevant alternatives is called Smith-IIA.

[7] See ref. [4]; for the proof that a tournament graph is irreducible if and only if it is strongly connected see Section 3, pp. 6–7.

[8] For a proof that a tournament graph is strongly connected if and only if it contains a Hamiltonian cycle see https:\\math.stackexchange.com\questions\191943\prove-that-strongly-connected-tournament-has-a-hamiltonian-cycle.

[9] The Online Encyclopedia of Integer Sequences, available at https:\\oeis.org\A051337.

[10] See ref. [4]; for a tabulation of this probability up to N=16 see Table 1 on p. 3; for a formula giving a bound for $N \geq 2$ see Theorem 1 on p. 4.

[11] Software is Maplesoft Maple™ 2001 running on a stock, Dell Inc. laptop, system model XPS 15 9550, which is an xd64-based personal computer (PC) which has as the central processing unit (CPU) the Intel Core™ i7-6700HQ, which has a clock frequency of 2.60 GHz and has 4 Cores, and 8 Logical Processors. Available memory is 16.0 GB of Installed Physical Memory (RAM), and 34.9 GB of Total Virtual Memory.

[12] Given any two irreducible tournament graphs with N nodes that are not isomorphic, must this succeed in distinguishing them were the power q applied to the adjacency matrix sufficiently large? Probably not; for a discussion of this point see https:\\math.stackexchange.com\questions\101428\testing-graph-isomorphism-with-powers-of-the-adjacency-matrix.

[13] For a discussion of local stability and related concepts see for example C.T. Munger, Jr. *Not Instant Runoff but Ranked Pairs*, papers A–D, in particular paper C, section II F, pp. 7–10.

[14] For a discussion of inversion symmetry and related concepts see for example [13], paper C, section II G, p. 10. For the proof that Beatpath has inversion symmetry, see [3], Section 4.5, pp. 215–216 (where the identical concept is called reversal symmetry). A.M.D.G.

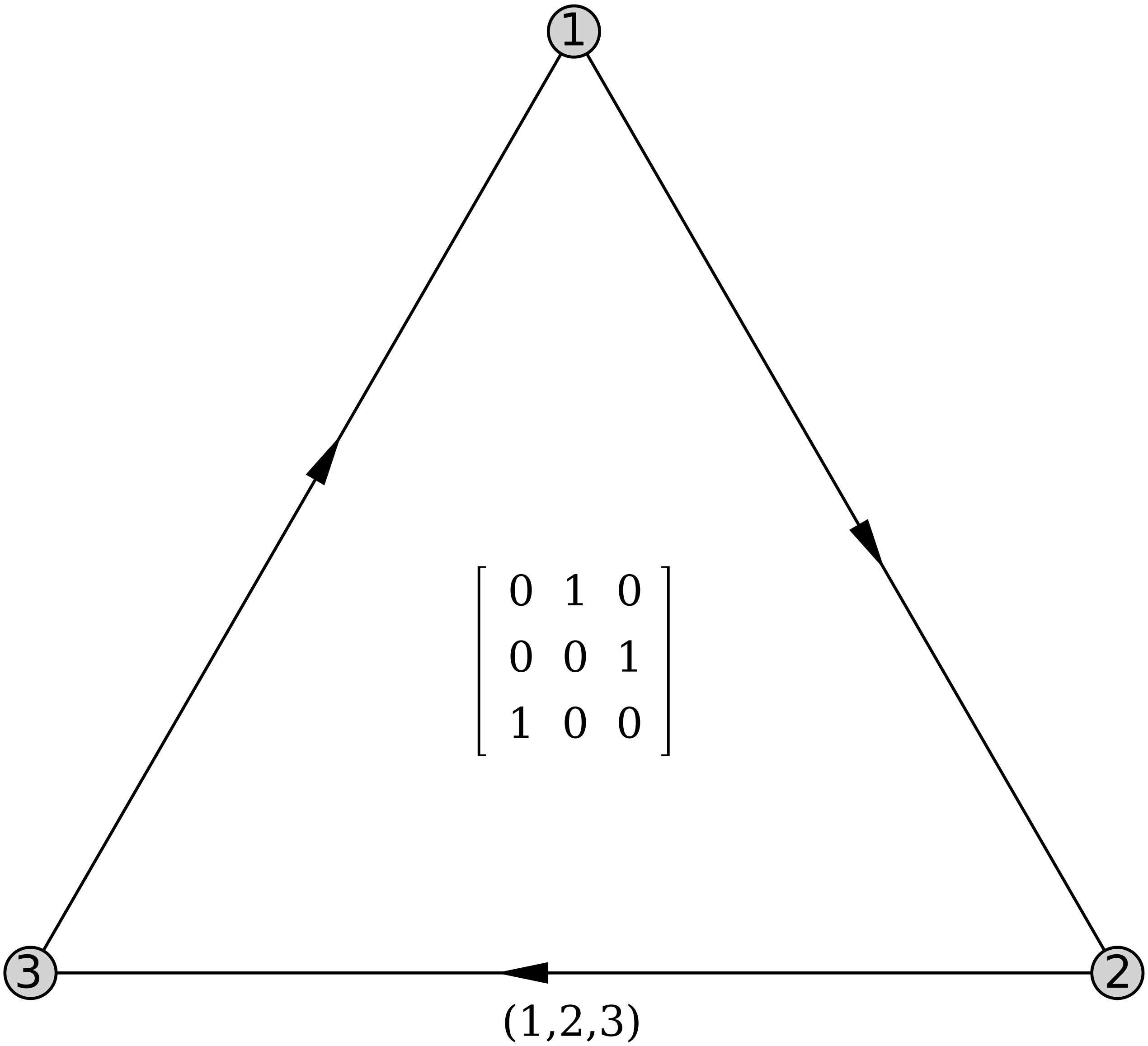

FIG. 1. The only cyclic tournament graph $N = 3$ candidates, up to a permutation of the candidates. The adjacency matrix for this graph is displayed. This graph as but the one Hamiltonian cycle 1231.

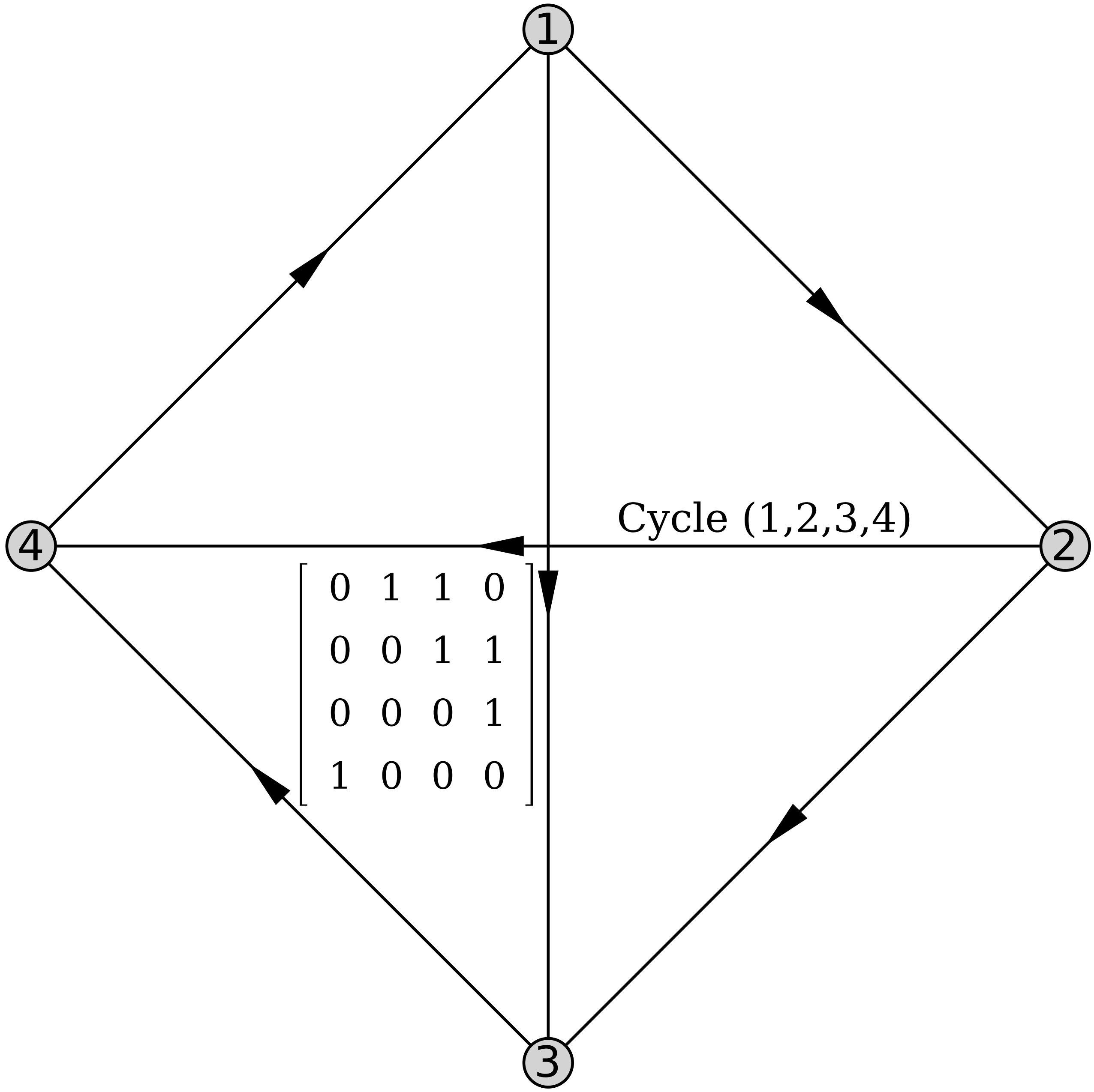

FIG. 2. The only cyclic tournament graph for $N = 4$ candidates, up to a permutation of the candidates. The adjacency matrix for this graph is displayed. This graph has but the one Hamiltonian cycle 12341.

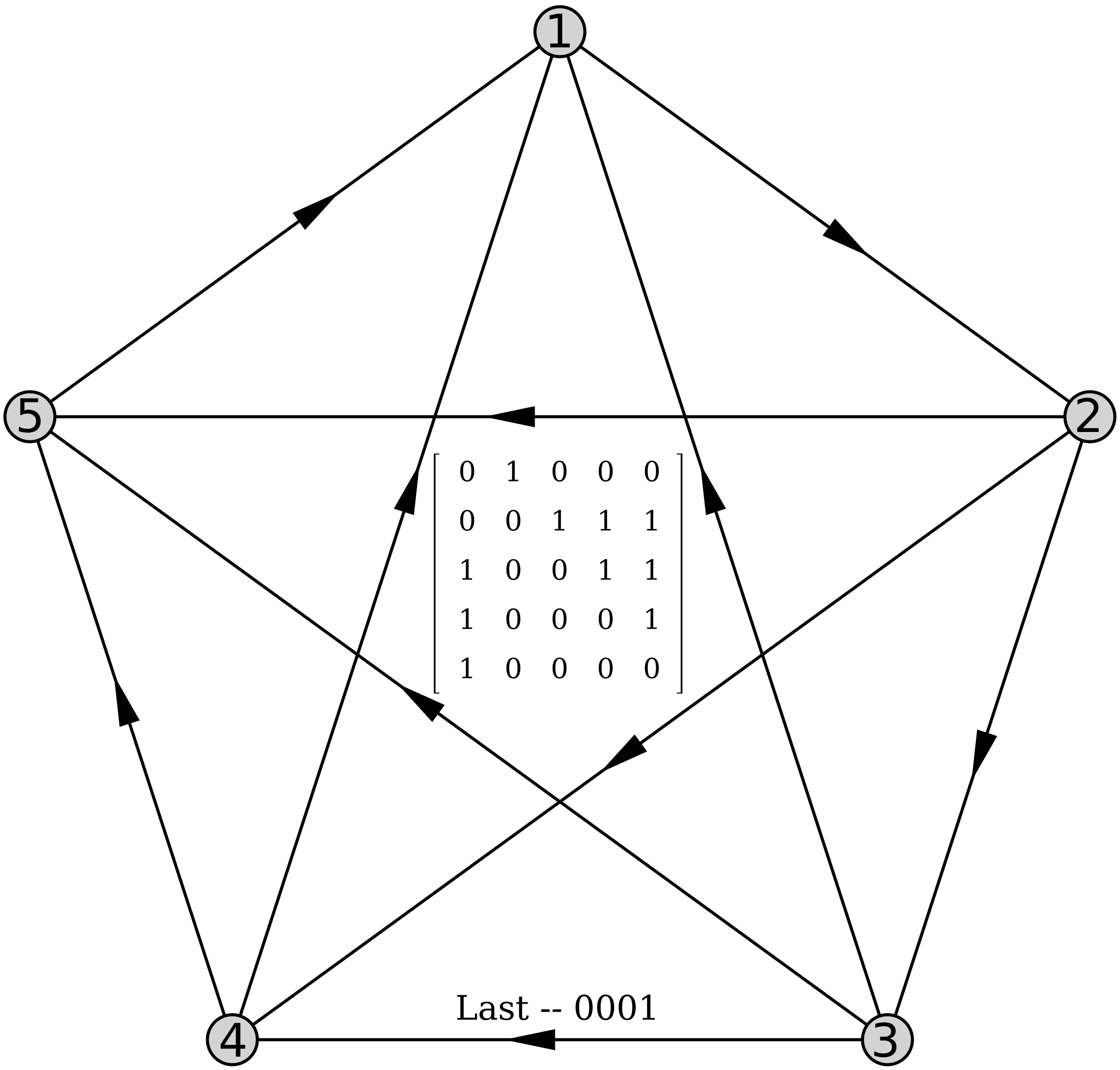

FIG. 3. One of the six cyclic tournament graphs for $N = 5$ candidates All six contains the clockwise 5-cycle on the outer pentagon of $1 \to 2 \to 3 \to 4 \to 5 \to 1$, as well as the horizontal link $2 \to 5$. They therefore differ only in their various orientations for the links $(1, 4)$, $(5, 3)$, $(4, 2)$, $(3, 1)$; if a link is anticlockwise like the link $2 \to 5$ we indicate it with a 1, otherwise with a 0; by that code, this is the 5-cycle 0001. The adjacency matrix for this graph is displayed. This graph has only the one Hamiltonian cycle 123451.

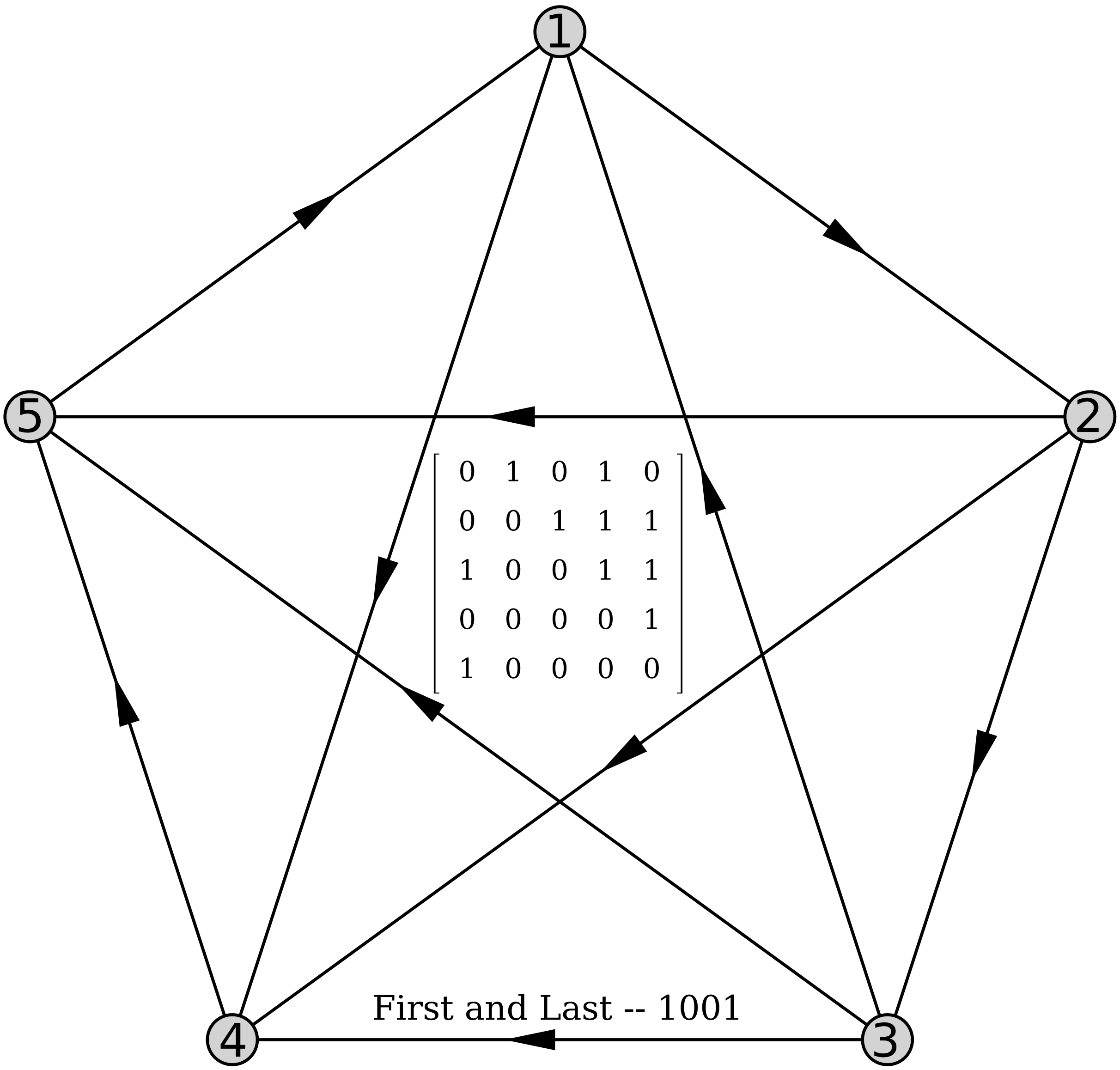

FIG. 4. One of the six cyclic tournament graphs for $N = 5$ candidates All six contains the clockwise 5-cycle on the outer pentagon of $1 \to 2 \to 3 \to 4 \to 5 \to 1$, as well as the horizontal link $2 \to 5$. They therefore differ only in their various orientations for the links $(1, 4)$, $(5, 3)$, $(4, 2)$, $(3, 1)$; if a link is anticlockwise like the link $2 \to 5$ we indicate it with a 1, otherwise with a 0; by that code, this is the 5-cycle 1001. The adjacency matrix for this graph is displayed. This graph has only the one Hamiltonian cycle 123451.

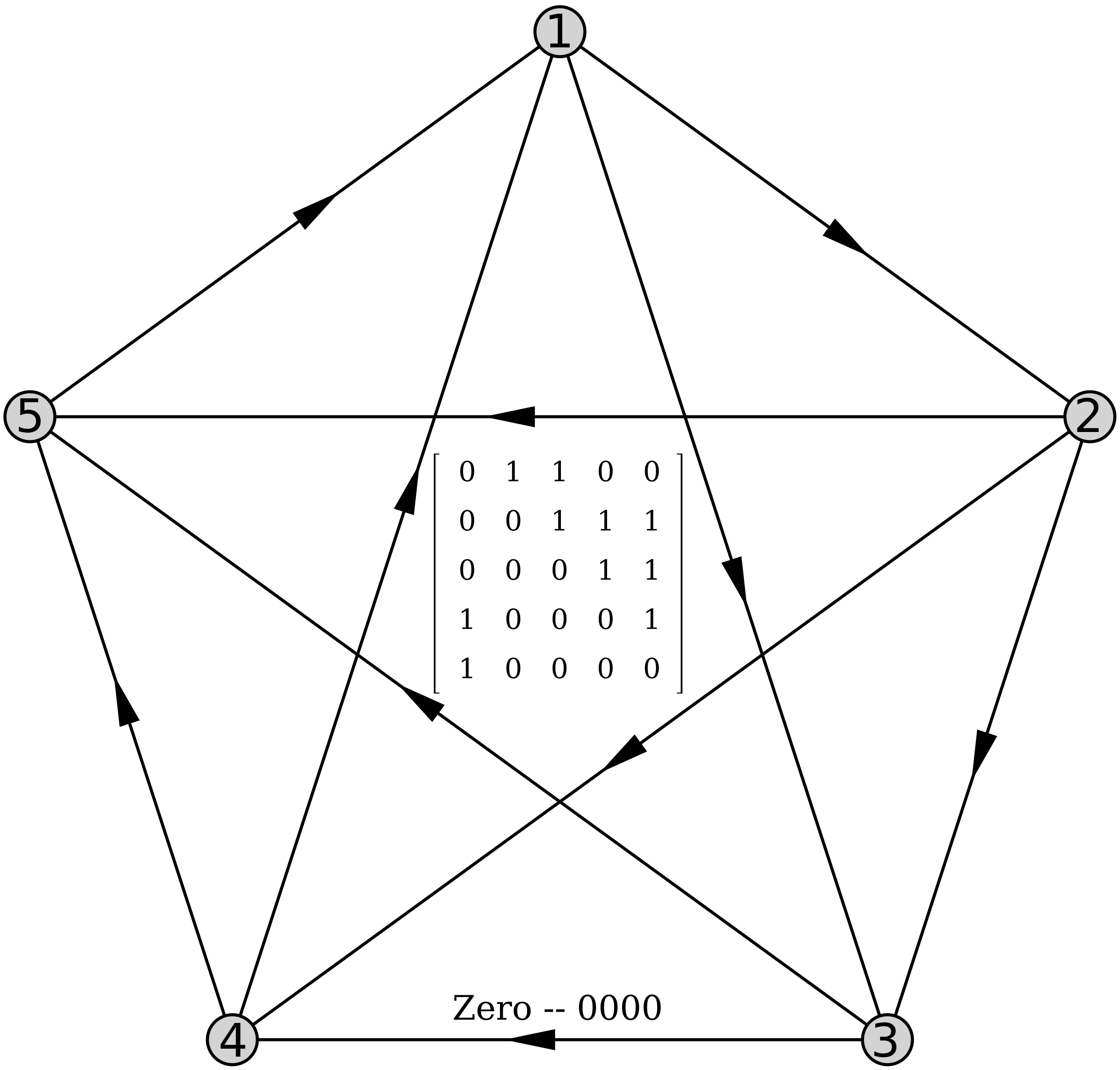

FIG. 5. One of the six cyclic tournament graphs for $N = 5$ candidates All six contains the clockwise 5-cycle on the outer pentagon of $1 \to 2 \to 3 \to 4 \to 5 \to 1$, as well as the horizontal link $2 \to 5$. They therefore differ only in their various orientations for the links $(1, 4)$, $(5, 3)$, $(4, 2)$, $(3, 1)$; if a link is anticlockwise like the link $2 \to 5$ we indicate it with a 1, otherwise with a 0; by that code, this is the 5-cycle 0000. The adjacency matrix for this graph is displayed. This graph has only the one Hamiltonian cycle 123451.

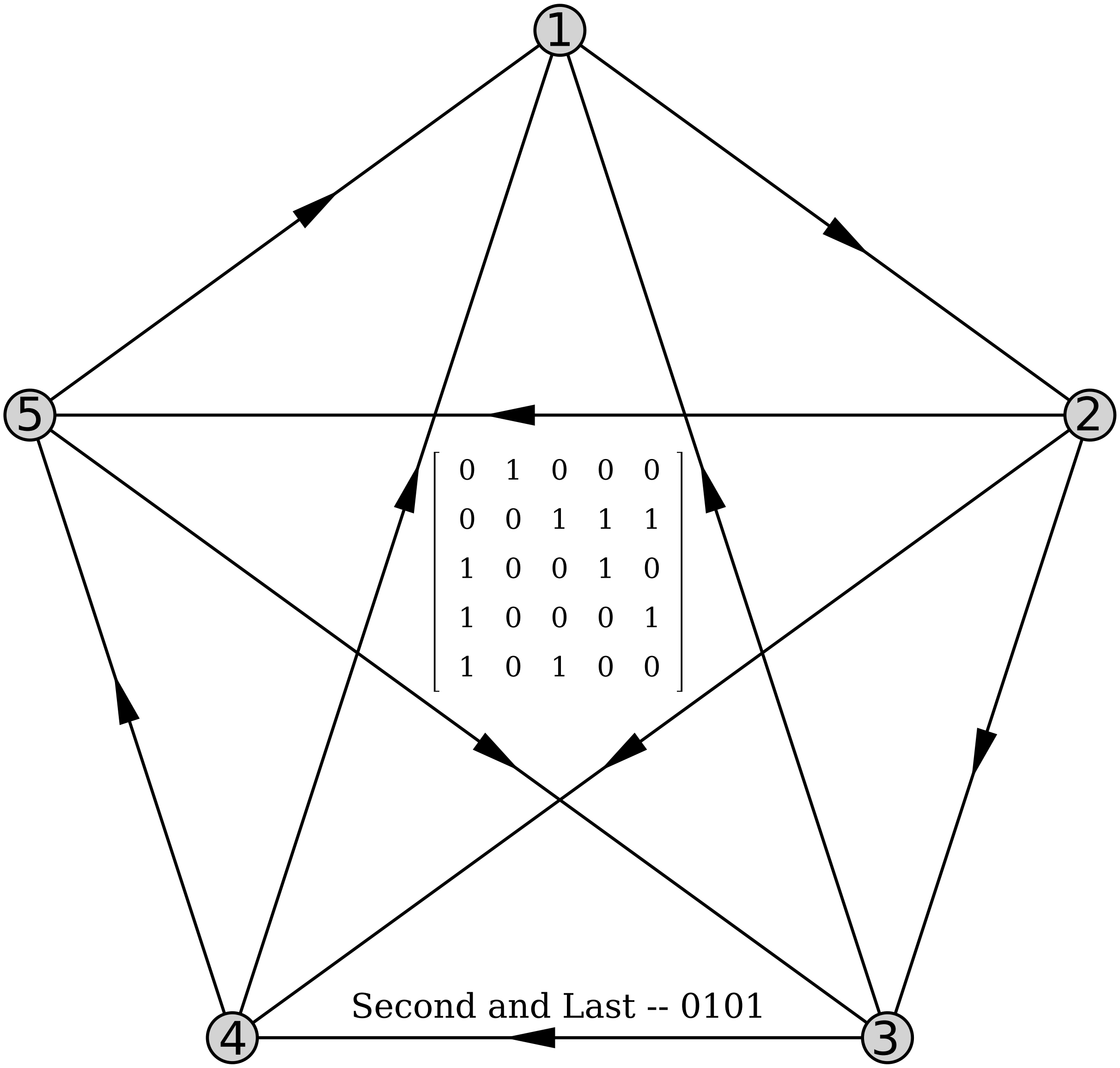

FIG. 6. One of the six cyclic tournament graphs for $N = 5$ candidates All six contains the clockwise 5-cycle on the outer pentagon of $1 \to 2 \to 3 \to 4 \to 5 \to 1$, as well as the horizontal link $2 \to 5$. They therefore differ only in their various orientations for the links $(1, 4)$, $(5, 3)$, $(4, 2)$, $(3, 1)$; if a link is anticlockwise like the link $2 \to 5$ we indicate it with a 1, otherwise with a 0; by that code, this is the 5-cycle 0101. The adjacency matrix for this graph is displayed. This graph has just two Hamiltonian cycles, 123451 and 125341.

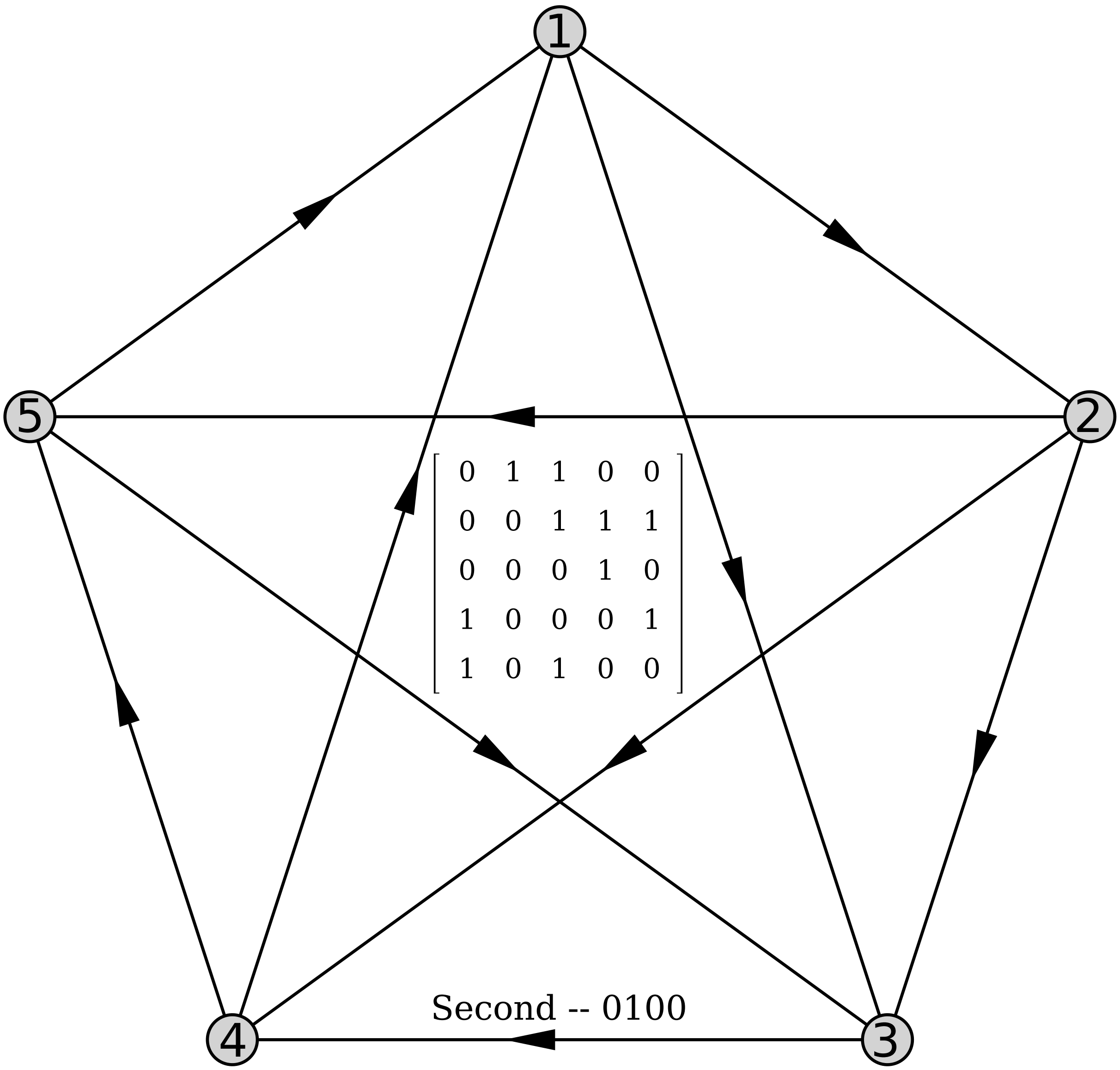

FIG. 7. One of the six cyclic tournament graphs for $N = 5$ candidates All six contains the clockwise 5-cycle on the outer pentagon of $1 \to 2 \to 3 \to 4 \to 5 \to 1$, as well as the horizontal link $2 \to 5$. They therefore differ only in their various orientations for the links $(1, 4)$, $(5, 3)$, $(4, 2)$, $(3, 1)$; if a link is anticlockwise like the link $2 \to 5$ we indicate it with a 1, otherwise with a 0; by that code, this is the 5-cycle 0100. The adjacency matrix for this graph is displayed. This graph as just two Hamiltonian cycles, 123451 and 125341

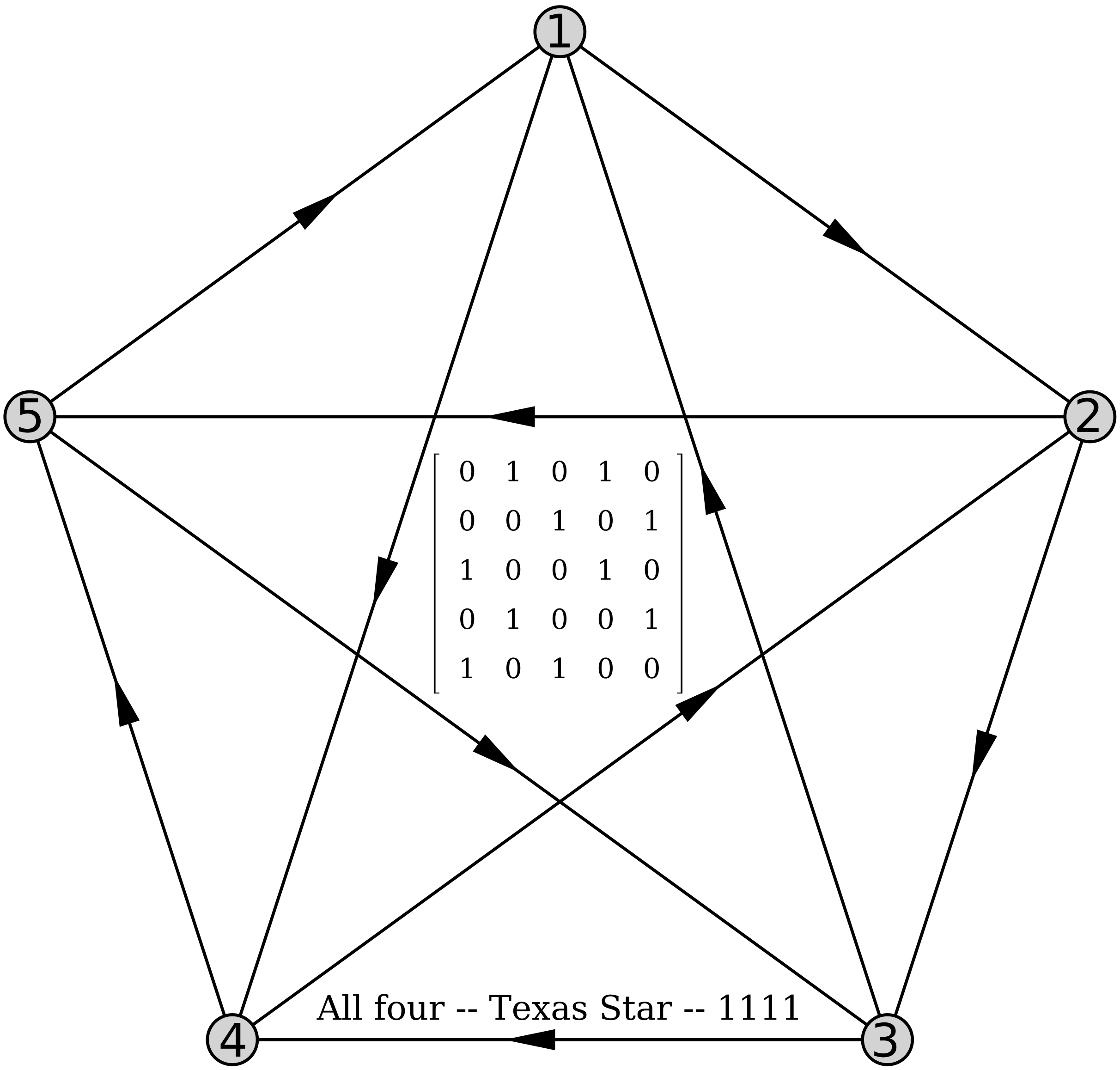

FIG. 8. One of the six cyclic tournament graphs for $N = 5$ candidates All six contains the clockwise 5-cycle on the outer pentagon of $1 \to 2 \to 3 \to 4 \to 5 \to 1$, as well as the horizontal link $2 \to 5$. They therefore differ only in their various orientations for the links $(1, 4)$, $(5, 3)$, $(4, 2)$, $(3, 1)$; if a link is anticlockwise like the link $2 \to 5$ we indicate it with a 1, otherwise with a 0; by that code, this is the 5-cycle 1111. The adjacency matrix for this graph is displayed. This graph as just two Hamiltonian cycles, 123451 and 142531.